Mechanics & Relativity

MG15

Contents/plan:

- §1 — overview of part I
- §2 — energy & momentum: m; phot.
- §3 — conservation of 4-momentum
- §4 — collision problems
- §5 — acceleration & force
- $\boxed{§6}$ — 4-vector formalism

After: problems ⊕ discussion.

special cases

§1.

2 postulates:

1. principle Einstein: laws of physics are invariant under space and time transfs.
2. max velocity same in all inertial frames (c)

3 effects:

1. loss of simult.
2. time is dilated $t = \gamma t'$ — observe more time passes.
3. length contra. $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$, $L = \frac{1}{\gamma} L'$ — observe shorter length

Lorentz trfs — LINEAR — Rapidity — MORIN — SYMM

$S' = S$ interval invariant

$S = c^2t^2 - x^2$

Invariant

S frame: $(ct, \vec{x}) = (ct, x, y, z)$

S' frame: $(ct', \vec{x}')$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right)$$
$$x' = \gamma(x - Vt)$$
$$y' = y$$
$$z' = z$$

Lorentz Transformation

$$dt' = \gamma\left(dt - \frac{V}{c^2}dx\right)$$
$$dx' = \gamma(dx - V\,dt)$$
$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - V dt)}{\gamma(dt - \frac{V}{c^2}dx)} \quad \Big| \frac{1}{dt}$$
$$= \frac{v_x - V}{1 - \frac{V}{c^2}v_x}$$
$$v'_y = \frac{1}{\gamma}\frac{v_y}{1 - \frac{V}{c^2}v_x}$$
$$v'_z = \frac{1}{\gamma}\frac{v_z}{1 - \frac{V}{c^2}v_z}$$

Einstein Trfs

§2.

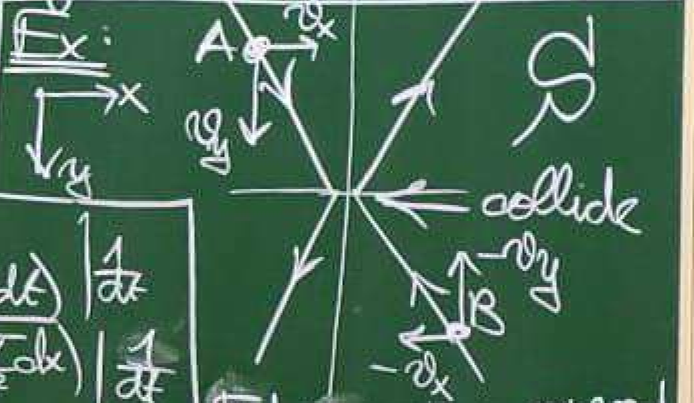
Before: $m_1 \; v_1 \rightarrow \quad \leftarrow v_2 \; m_1$

After: $v_{aft} \; m_1 \; m_2 \; v_{aft}$

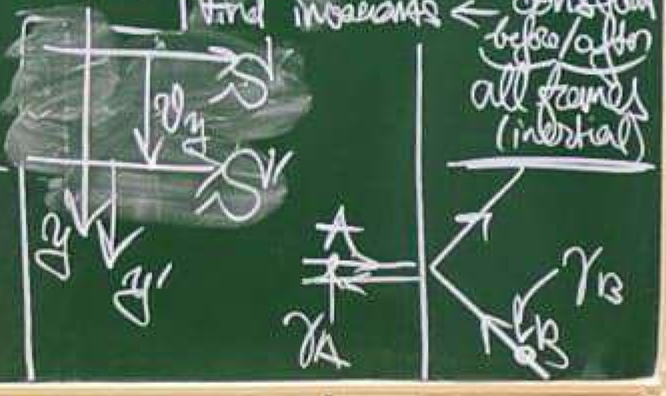
Invariant: energy, momentum — all inertial frames

$E_{class} = \frac{1}{2}mv^2$; $\vec{p}_{clas} = m\vec{v}$

Ex:



Find invariants ← consequences before/after all frames (inertial)



$$v'_{xA} = \frac{1}{\gamma}\frac{v_x}{1 - v_y^2/c^2}$$
$$v'_{xB} = \frac{1}{\gamma}\frac{(-)v_x}{1 - \frac{v_y}{c^2}(-)v_y} = \frac{1}{\gamma}\frac{(-)v_x}{1 + v_y^2/c^2}$$
$$v'_{yA} = 0$$
$$v'_{yB} = \frac{-v_y - v_y}{1 + v_y^2/c^2}$$

x-comp of "momentum" $\vec{p} = \alpha m\vec{v}$ — α function of v

$\boxed{p'_{xA} = p'_{xB}}$

$$\alpha_A m v'_{xA} = \alpha_B m v'_{xB} \rightarrow \frac{\alpha_B}{\alpha_A} = \frac{1 + v_y^2/c^2}{1 - v_y^2/c^2} = \frac{\gamma_B}{\gamma_A}$$

Limit $v_x \rightarrow 0$

$\gamma_A \rightarrow 1$; $\vec{p} = \vec{p}_{clas}$

$\alpha_A m\vec{v} = m\vec{v}$

$\alpha_B = 1 \cdot \frac{\gamma_B}{1} = \gamma_B$, $\alpha_A = 1$

$$\vec{p} = \gamma m\vec{v}$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$E = \gamma mc^2$ ← Role

S frame: E, $\vec{p}$

S' frame: E', $\vec{p}'$

$m \; \vec{v}$

$$p = \gamma m v = \frac{mv}{\sqrt{1 - v^2/c^2}}$$

$E' = \gamma_V (E - V p_x)$

$p'_x = \gamma_V \left(p_x - \frac{V}{c^2} E\right)$

$p'_y = p_y$

$p'_z = p_z$

Transforms via Lorentz ⇒ interval?

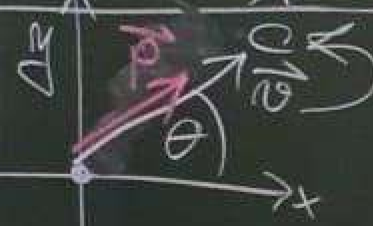
$\frac{E^2}{c^2} - p^2 = m^2c^2$ Invariant

limit $m \to 0$: $\frac{E^2}{c^2} - p^2 \to 0$; $\frac{E^2}{c^2} = p^2$; $E>0, p>0$

Very important deal ; $p = \sqrt{p_x^2+p_y^2+p_z^2}$

... for a photon wavelength → $p = \frac{E}{c}$

$E = h\frac{c}{\lambda}$; h - Plancks (proportionality)



$v = c$

$\vec v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix} = c\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\vec p = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} p\cos\theta \\ p\sin\theta \end{pmatrix} = \frac{E}{c}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

§3) By construction: $E, \vec p$ conserved

before: m_1 v_1 → ← v_2 m_2 ; v_2 magn.

after: c ← E ~ M → u

① Conservation of energy:

$E^{before} = E^{after}$; kg $\frac{m^2}{s^2}$

$E_1 + E_2 = E_M + E$

$\gamma_1 mc^2 + \gamma_2 mc^2 = \gamma_u Mc^2 + E$

② Conservation of momentum:

x-dir: $p_{1x} + p_{2x} = p_{Mx} + p_{photon,x}$

$\gamma_1 m v_1 - \gamma_2 m v_2 = \gamma_u M u - \frac{E}{c}$

y-dir: $0 + 0 = 0 + \frac{E}{c}\sin\theta$; $\sin\theta = 0$; $\theta = 0$

conservation of 4-momentum

$\boxed{(E/c, \vec p)}$ ← each of the 4 components conserved.

§5. Acceleration & Force ; $F = \frac{dE}{dx}$

$\vec F = \frac{d\vec p}{dt} = \frac{d}{dt}(\gamma m \vec v)$; a

1-dim case p ← only in $\hat x$

$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = m\left(\frac{d\gamma}{dt} v + \gamma\frac{dv}{dt}\right)$

$= \gamma^3 m a$

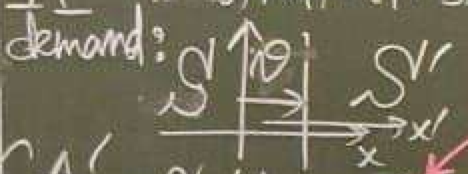
2-dim $F_x = \gamma^3 m a_x$; $F_y = \gamma m a_y$

$\gamma > 1 \Rightarrow \gamma^3 > \gamma \Rightarrow \boxed{F_x > F_y \text{ for } a_x = a_y}$

§6 4-vectors

vector with 4 comps: $A = (A_0, A_1, A_2, A_3)$

demand:



SR units

$A'_0 = \gamma_v (A_0 - v A_1)$

$A'_1 = \gamma_v (A_1 - v A_0)$

$A'_2 = A_2$

$A'_3 = A_3$

$\boxed{A \text{ 4-vector}}$

NB! Relativity does not live in Euclidean geometry

Geo: [Minkowski]

$A\cdot B = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3$; (B_0, B_1, B_2, B_3)

$A\cdot A = A_0^2 - A_1^2 - A_2^2 - A_3^2$

$R\cdot R = c^2t^2 - x^2$ Inv!

$P\cdot P = \left(\frac{E}{c}\right)^2 - p^2$

Show: $A\cdot B$ invariant under Lorentz.

Ex: pos $R = (ct, \vec x)$

velocity $V = \frac{dR}{d\tau} = \gamma\frac{dR}{dt} = \gamma(c, \vec v)$

$d\tau^2 = dt^2 - d\vec x^2/c^2$

$d\tau = \frac{1}{\gamma} dt$

4-momentum $P = mV = (\gamma mc, \gamma m\vec v)$; $\frac{\gamma mc^2}{c}$

acceleration $A = \frac{dV}{d\tau}$ works

force $F = mA = \frac{dP}{d\tau}$

Remark: Lorentz transformations describe symmetry in geometry

m → v ; S

S' m ← $P'\cdot P' = \frac{E'^2}{c^2} = m^2c^2$

$E' = mc^2$

$E = mc^2$; $P\cdot P = P'\cdot P'$

$\frac{E^2}{c^2} - p^2 = m^2c^2$

§4. Collisions: What to do?

① find 4-momenta of all particles

$P_m = (\gamma mc, \gamma m v\cos\theta, \gamma m v\sin\theta, 0)$

$P_{phot} = \left(\frac{E}{c}, \frac{E}{c}\cos\theta, \frac{E}{c}\sin\theta, 0\right)$

may be unknown vs known

② 4-mom conserved ⇒ 3 equations (≤ 3 unknowns ⇒ Can solve)

③ use inner product of 4-vectors

Ex: V | S ; v_2 m_2 ; S' ; W

m_1 ; (rest)

$V_1^{(S)} \cdot V_2^{(S)} = V_1^{(S')} \cdot V_2^{(S')}$; known $\gamma(c, 0)$; unknown → SOLVE

④ once you know P: mass

W ; m

$\boxed{P\cdot P = m^2c^2}$ ← 1 unknown

~~> $\boxed{P_{ph}\cdot P_{ph} = 0}$