MECHANICS & RELATIVITY:
FRANCKEN PRACTICE SESSION PROBLEMS

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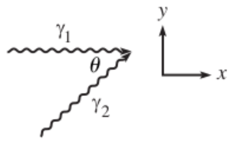
Recall some useful formulae:

(1) for massive ($m > 0$) particles energy $E = \gamma mc^2$ and momentum $\vec{p} = \gamma m\vec{v}$;
(2) for photons $p = E/c$; if p in the $+\hat{x}$ direction, $P = (E/c, E/c, 0, 0)$;
(3) 4-momentum is defined as $P = (E/c, \vec{p})$; note: capital P for the 4-vector, small $\vec{p}$ with a vector arrow for the momentum vector; magnitude $||\vec{p}|| = p$;
(4) very important relation $m^2c^2 = P \cdot P = E^2/c^2 - p^2$;
(5) 4-momentum is conserved in collisions – energy $E^{\text{before}} = E^{\text{after}}$ and momentum $p_x^{\text{before}} = p_x^{\text{after}}$, $p_y^{\text{before}} = p_y^{\text{after}}$, $p_z^{\text{before}} = p_z^{\text{after}}$ are conserved;
(6) the inner product $V_1^{\text{lab}} \cdot V_2^{\text{lab}} = V_1^{\text{other}} \cdot V_2^{\text{other}}$ is frame-invariant.

Problem 1. Two photons collide at an angle θ as measured in the lab frame:



Upon colliding, the photons create a particle of mass M. Let both photons have the same energy E.

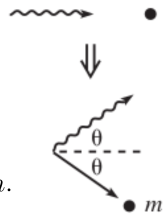
(a) Find M in terms of E and θ.

(b) What is the magnitude v of the velocity $\vec{v} = (v_x, v_y, 0)$ of the particle? What the the components v_x and v_y? Check that $v^2 = v_x^2 + v_y^2$.

Hint (a & b): Conservation of 4-momentum; then use the very-important relation for P of M, namely, $M^2c^2 = P \cdot P$.

Answers: $M = (E/c^2)\sqrt{2(1 - \cos\theta)}$, $v = c\sqrt{(1 + \cos\theta)/2}$, $v_x = c(1 + \cos\theta)/2$, $v_y = c(\sin\theta)/2$.

Problem 2. A photon of energy E (in the lab frame) travelling in the $+\hat{x}$ direction hits a particle of mass m. After the collision both the photon (note: different energy!) and mass travel at an angle θ from the x-axis. Denote the energy of the photon after collision by E' (in the lab frame).



(a) Find θ in terms of E and m. *Hint:* Find P_m^{after} of m; use $P \cdot P$.

Answer:

$$\cos\theta = \frac{E + mc^2}{E + 2mc^2} \quad \Longleftrightarrow \quad \theta = \arccos\frac{E + mc^2}{E + 2mc^2}$$

(b) In the limit of a heavy particle and low-energy photon $E \ll mc^2$, what is θ?

(c) Find E' in terms of E and m (use the expression of θ).

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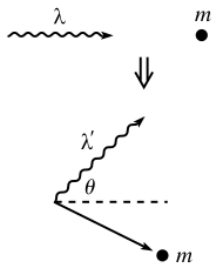
Problem 3. Let a particle of mass m be stationary in the lab frame; consider a photon travelling in the $+\hat{x}$ direction that collides with m. Let the photon have wavelength λ (hence energy $E = hc/\lambda$, where h is the Planck constant). After collision the photon scatters at an angle θ and has wavelength λ' as given below



Show that

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta).$$

Hint: Express energies in terms of the wavelengths. Conservation of 4-momentum; use $m^2c^2 = P_m^{\text{after}} \cdot P_m^{\text{after}}$, where express P_m^{after} in terms of P_m^{before}, $P_{\text{photon}}^{\text{before}}$, and $P_{\text{photon}}^{\text{after}}$. What are $P_m^{\text{before}} \cdot P_m^{\text{before}} =?$ and $P_{\text{photon}}^{\text{before}} \cdot P_{\text{photon}}^{\text{before}} = P_{\text{photon}}^{\text{after}} \cdot P_{\text{photon}}^{\text{after}} =?$

Problem 4. Consider a collection of n particles in space (3-dimensional space). Suppose that they have $\vec{p}_{\text{tot}} = \sum_{i=1}^n \vec{p}_i$ and $E_{\text{tot}} = \sum_{i=1}^n E_i$ as measured in the lab frame.

(a) First consider *classical mechanics*. Let the total mass be $M = \sum_{i=1}^n m_i$. Given the positions of the particles $\vec{r}_1, ..., \vec{r}_n$, what is the position $\vec{r}$ of the centre-of-mass in the lab frame? From this find the velocity v of the centre-of-mass in the lab frame in therms of $\vec{p}_{\text{tot}}$ and M!

Answer: $\vec{p}_{\text{tot}} = M\vec{v}$.

(b) Still in *classical mechanics*, if the total momentum $\vec{p}_{\text{tot}}$ in the lab frame is zero, is the lab frame the centre-of-mass frame?

(c) Now in relativity. We define the centre-of-mass frame as the frame, where the total momentum is zero. Given $\vec{p}_{\text{tot}}$ and E_{tot} (in the lab frame) find the velocity vector $\vec{v} = (v_x, v_y, v_z)$ (in the lab frame) of the centre-of-mass frame.

Hint: Use a Lorentz transformation from the lab into the centre-of-mass* frame; to find v_x boost from lab to c.-of-m. in the x direction; similarly for finding v_y, v_z.

Answer: $\vec{v} = (c^2/E_{\text{tot}})\vec{p}_{\text{tot}}$.

Problem 5. Consider two particles of mass m; in the lab frame one particle (1) travelling with velocity $(3/5)c$ hits the other mass (2), which is stationary:

$$_{(1)}\ m \xrightarrow{3c/5} \qquad\qquad m\ _{(2)}$$

(a) What are the energies E_1, E_2 and momenta p_1, p_2 of the two particles in the lab frame?

(b) Find the velocity v (in the lab frame) of the centre-of-mass frame!

Hint: Use Problem 4(c). *Answer:* $v = (1/3)c$.

(c) What are the energies E_1', E_2' and momenta p_1', p_2' of the two particles in the centre-of-mass frame?

Answer: For the first particle $E_1 = (3/2\sqrt{2})mc^2$, $p_1 = (1/2\sqrt{2})mc$.

Problem 6. We will work in the lab frame. Consider a mass m that starts at rest at $x = 0$; we push on the mass with a constant force F in the $+x$-direction.

(a) Find the time t it takes for the mass to reach position x ($x > 0$).

Hint: $F = dp/dt = \text{const}$, $F = dE/dx = \text{const}$, therefore... (1st order ODEs). Then use the very important relation $m^2c^2 = E^2/c^2 - p^2$.

Answer: $t = \sqrt{2mFx + F^2x^2/c^2}/F$

(b) Simplify the answer in the limit $1/x \approx 0$ using the binomial approximation.

Problem 7. Show that the proper time interval $d\tau = \sqrt{dt^2 - dr^2/c^2}$ frame-invariant, where $dr^2 = dx^2 + dy^2 + dz^2$. (I.e. $d\tau$ does not change after a Lorentz transformation.)

Problem 8. Recall that the force 4-vector F is defined as $F = dP/d\tau$.

(a) Show that

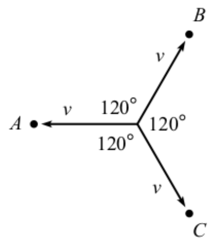
$$F = \gamma \left(\frac{1}{c}\frac{dE}{dt}, \vec{f} \right),$$

where $\vec{f} = d\vec{p}/dt = d(\gamma m \vec{v})/dt$ is the force vector.

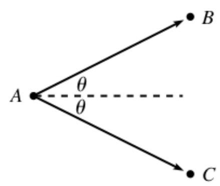
(b) Show that if the mass m is constant ($dm/dt = 0$), then $F = mA$, where A is the acceleration 4-vector $A = dV/d\tau$, $V = (dt/d\tau, dx/d\tau, dy/d\tau, dz/d\tau)$.

(c) In what frame is $\vec{f} = m\vec{a}$?

Problem 9. In the lab frame a particle decays, creating three particles of the same mass and equal velocity v travelling symmetrically in 120° angles:



In the frame of particle A, let the angle between particles B and C be θ, as given below:



(a) Find the velocity 4-vectors $V = \gamma(c, \vec{v})$ of each particle in the lab frame; use the notation V_A^{lab}, V_B^{lab}, and V_C^{lab}.

(b) Compute the inner products $V_A^{\text{lab}} \cdot V_B^{\text{lab}}$, $V_A^{\text{lab}} \cdot V_C^{\text{lab}}$, and $V_B^{\text{lab}} \cdot V_C^{\text{lab}}$. Why are all of them equal?

Answer: $\gamma^2 c^2 (1 + v^2/(2c^2))$.

(c) Compute the velocity 4-vectors of each particle in the frame of particle A, denoted $V_A^{(A)}$, $V_B^{(A)}$, and $V_C^{(A)}$.

(d) Using invariance of the inner product, find θ.

Hint: $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

Answer:

$$\cos 2\theta = \frac{1 + v^2/(2c^2)}{2 - v^2/(2c^2)} \quad \Longleftrightarrow \quad \theta = \frac{1}{2} \arccos \frac{1 + v^2/(2c^2)}{2 - v^2/(2c^2)}$$

(e) What does θ reduce to in the limits $v \approx 0$ and $v \approx c$?