ELECTRICITY
&
MAGNETISM

Supplement and recapitulatory notes

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CONTENTS

CHAPTER I

MATHEMATICAL PRELIMINARIES

§1. **Recapitulation of vector calculus**

Differentiation rules for vector functions may be found on the first/last cover page of Griffiths and will be often used:

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1.1}$$

$$\nabla(fg) = f\nabla g + g\nabla f \tag{1.2}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \tag{1.3}$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f \tag{1.4}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{1.5}$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} - \mathbf{A} \times \nabla f \tag{1.6}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \tag{1.7}$$

$$\nabla \times \nabla f = 0 \tag{1.8}$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{1.9}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{1.10}$$

1.1 Remark. Note that $(\mathbf{A} \cdot \nabla)\mathbf{B}$ is read as first taking the dot product of $\mathbf{A}$ with ∇, obtaining an operator $\mathbb{R}^3 \to \mathbb{R}^3$ on the vector $\mathbf{B}$:

$$(\mathbf{A} \cdot \nabla)\mathbf{B} = \left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right)\mathbf{B}.$$

In spherical or cylindrical coordinates the gradient, divergence, and curl must be expressed in the coordinates using partial derivatives; the $\hat{x}$ component: $\partial f(r, \phi, \theta)/\partial x = (\partial f/\partial r)(\partial r/\partial x)$ using the chain rule. NB! the second partial term is not identity. And then substitute the x, y, z partials into the expressions for r, ϕ, θ directions:

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi \sin\theta & \sin\phi \sin\theta & \cos\theta \\ \cos\phi \cos\theta & \sin\phi \cos\theta & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}. \tag{1.11}$$

Line and surface integrals are evaluated as follows: let $\gamma(t) : \mathbb{R} \to \mathbb{R}^3$ be a parametrisation of the line C and $\gamma(s, t)$ a parametrisation of the surface S; then

$$\int_C f d\ell = \int_0^1 f(\gamma(t))|\gamma'(t)|\, dt, \qquad \int_S f da = \int_0^1 \int_0^1 f(\gamma(s,t)) \left| \frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t} \right| ds\, dt,$$

$$\int_C \mathbf{F} \cdot d\boldsymbol{\ell} = \int_0^1 \mathbf{F}(\gamma(t)) \cdot \gamma'(t)\, dt, \qquad \int_S \mathbf{F} \cdot d\mathbf{a} = \int_0^1 \int_0^1 \mathbf{F}(\gamma(s,t)) \cdot \left(\frac{\partial \gamma}{\partial s} \times \frac{\partial \gamma}{\partial t} \right) ds\, dt.$$

Upon changing coordinates to polar/cyllindrical or spherical, the differential term must be multiplied by a modulus of the Jacobian: r for cyllindrical, $r^2 \sin\theta$ for spherical.

§2. **Important theorems**

2.1 Theorem (Fundamental theorem of calculus). *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that the derivative f' is Riemann-integrable. Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{2.1}$$

2.2 Remark. The condition that the derivative is integrable is necessary! There exist differentiable functions whose derivative is not integrable; these are, however, quite pathological.

Proof. Let $\epsilon > 0$. Since f' is integrable, there exists a partition P_ϵ of the interval $[a,b]$ into smaller subintervals $P_\epsilon = \{[a = x_0, x_1], [x_1, x_2], ..., [x_{n-1}, x_n = b]\}$, such that the upper and lower Riemann sums

$$U(P_\epsilon, f) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot \sup_{x\in[x_{k-1},x_k]} f(x)$$
$$L(P_\epsilon, f) = \sum_{k=1}^{n} (x_k - x_{k-1}) \cdot \inf_{x\in[x_{k-1},x_k]} f(x)$$

differ by less than epsilon: $U(P_\epsilon, f)-L(P_\epsilon, f) < \epsilon$; since the integral is the smallest (infimum) possible upper sum and largest lower sum, we have $L(P_\epsilon, f) \leq \int_a^b f(x)dx \leq U(P_\epsilon, f)$. Now, letting $s_k \in [x_{k-1}, x_k]$ be arbitrary, we have $\inf_{x\in[x_{k-1},x_k]} f(x) \leq f(s_k) \leq \sup_{x\in[x_{k-1},x_k]} f(x)$, thus it follows that $L(P_\epsilon, f) \leq \sum_{k=1}^{n} f(s_k)(x_k - x_{k-1}) \leq U(P_\epsilon, f)$. Finally, taking the difference of the two inequalities, we obtain

$$\left|\sum_{k=1}^{n} f(s_k)(x_k - x_{k-1}) - \int_a^b f(x)dx\right| < \epsilon. \tag{2.2}$$

All that is left to do is to supply s_k. By the mean value theorem on any interval $[x_{k-1}, x_k]$ there exists $t_k \in (x_{k-1}, x_k)$ such that $f(x_k) - f(x_{k-1}) = f'(t_k)(x_k - x_{k-1})$; this gives a telescoping (adjacent terms cancel each other) finite[1] sum

$$\sum_{k=1}^{n} f'(t_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(b) - f(a).$$

Therefore taking $s_k = t_k$, it follows by (2.2) that $\left|f(b) - f(a) - \int_a^b f(x)\,dx\right| < \epsilon$ for any arbitrarily small ϵ; it follows (e.g. by limiting $\epsilon = 1/n \xrightarrow{n\to\infty} 0$) that the difference is zero, hence $f(b) - f(a) = \int_a^b f(x)\,dx$. □

2.3 Remark. The theorem generalises to derivatives (gradients[2]) of functions $\mathbb{R}^m \to \mathbb{R}$ (typically $m = 3$ in physics) under line integrals: $\int_{\gamma:\mathbf{a}\to\mathbf{b}} \nabla f \cdot d\boldsymbol{\ell} = f(\mathbf{b}) - f(\mathbf{a})$.

We now present two theorems of multivariable calculus, the (intuitive) proofs of which may be found it Appendix A of Griffiths.

2.4 Theorem (Gauss's divergence theorem). *Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector function in a volume V and integrable over the closed boundary ∂V of the volume. Then*

$$\int_V (\nabla \cdot \mathbf{F})\,d\tau = \oint_{\partial V} \mathbf{F} \cdot d\mathbf{a}. \tag{2.3}$$

[1] For infinite sums absolute convergence is needed. Working with finite sums is identical to working with just a single + (sum).

[2] A gradient ∇f is a linear map of $\mathbb{R}^3$ to $\mathbb{R}$, namely, $(\nabla f)\mathbf{v} = (\nabla f) \cdot \mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3$.

2.5 Theorem (Stokes theorem). *Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable vector function on a surface S and integrable over the closed boundary ∂S of the surface. Then*

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \oint_{\partial S} \mathbf{F} \cdot d\boldsymbol{\ell}. \tag{2.4}$$

2.6 Remark. Stokes theorem generalises to functions $\mathbb{R}^n \to \mathbb{R}$. The details are found in multivariable analysis.

§3. **Formalisation of the Dirac delta**

3.1 Definition. Let $\alpha(x) : [a, b] \to \mathbb{R}$ be a monotonically increasing function (non-strictly, i.e. non-decreasing), let $\mathfrak{P}$ be the set of all possible[3] partitions P of $[a, b]$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Riemann-Stieltjes-α-integrable* (or the Riemann-Stieltjes integral $\int_a^b f(x)\, d\alpha(x)$ of f exists) if the upper and lower sums are equal (and we call that value the integral):

$$\inf_{P\in\mathfrak{P}} U(P, f, \alpha) = \sup_{P\in\mathfrak{P}} L(P, f, \alpha) \overset{\text{def}}{=}: \int_a^b f(x)\, d\alpha(x), \tag{3.1}$$

where the upper and lower sums of the partition $P = \{[a = x_0, x_1], ..., [x_{n-1}, x_n = b]\}$ are

$$U(P, f, \alpha) = \sum_{k=1}^{n} (\alpha(x_k) - \alpha(x_{k-1})) \sup_{x\in[x_{k-1},x_k]} f(x),$$
$$L(P, f, \alpha) = \sum_{k=1}^{n} (\alpha(x_k) - \alpha(x_{k-1})) \inf_{x\in[x_{k-1},x_k]} f(x).$$

3.2 Remark. We recover the Riemann integral as a special case: $\alpha(x) = x$.

3.3 Definition. Let $c \in (a, b)$. The Dirac delta $\delta(c)$ is an operator on a function $f : [a, b] \to \mathbb{R}$ that sends f to the Riemann-Stieltjes integral of f under the (Heaviside) step function:

$$\int_a^b \delta(x - c) f(x)\, dx :\overset{\text{def}}{=} \int_a^b f(x)\, d\Theta(x) = f(c), \tag{3.2}$$

where Θ is the step function

$$\Theta(x) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ 1 & \text{if} \quad x > 0 \end{cases}. \tag{3.3}$$

3.4 Remark. 1. Multiplying the operator by a constant yields the constant multiplied by the operator acting on the function; similarly for a sum.

2. For a comprehensive theory of generalised functions (of which the Dirac delta is an example) see I.M. Gel'fand, G.E. Shilov, *Generalised functions*, 1964, vol. I, *Properties and operators.*

§4. **Conservative vector fields**

4.1 Theorem (Helmholtz). *Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector function; let $\nabla \cdot \mathbf{F} = D$ and $\nabla \times \mathbf{F} = \mathbf{C}$ be known. If $\nabla \cdot \mathbf{C} = 0$, $\lim_{r\to\infty} r^2 D(\mathbf{r}) = 0$, and $\lim_{r\to\infty} r^2 \mathbf{C}(\mathbf{r}) = 0$, then the vector field may be (uniquely) broken down into two components via scalar and vector potentials:*

$$\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}, \tag{4.1}$$

where

$$U(\mathbf{r}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{D(\mathbf{r}')}{\mathscr{r}} d\tau' \qquad \textit{and} \qquad \mathbf{W}(\mathbf{r}) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{C}(\mathbf{r}')}{\mathscr{r}} d\tau', \tag{4.2}$$

with $\mathscr{r} := |\mathbf{r} - \mathbf{r}'|$.

[3]Formally, $\mathfrak{P} = \{\{[x_0, x_1], ..., [x_{n-1}, x_n]\} : a = x_0 < x_1 < \cdots < x_n = b\}$.

4.2 Remark. We will later see that for static electric fields the $\mathbf{W}$ curl component is zero, whereas for magnetic fields – the U potential component. Since for any vector function the divergence of the curl is zero, for the magnetic field $\mathbf{B} = \nabla \times \mathbf{W}_{\text{magn}}$, it follows that $\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{W}_{\text{magn}}) = 0$; similarly for the electric field: $\mathbf{E} = -\nabla U$, we have $\nabla \times \mathbf{E} = -\nabla \times \nabla U = 0$; these are two of the four Maxwell's equations (for fields that *do not* change in time[4]).

Proof. See Appendix B of Griffiths. □

[4]This already suggests what lies ahead...

CHAPTER II
ELECTROSTATICS

§5. **Electric field**

A charge radiates an electric field outward symmetrically in all directions; it must thus decrease as $1/S_{\text{sphere}} = 1/4\pi\mathscr{r}^2$ (surface area of a sphere) in all directions. The strength depends only on the magnitude of the charge, with a proportionality constant $1/\epsilon_0$, thus for a charge distribution (density) $\rho : \mathbb{R}^3 \to \mathbb{R}$ (may be 0 outside some fixed shape), the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'. \tag{5.1}$$

The statement above may indeed be formalised as the surface integral of the field along the normal of the surface.

5.1 Theorem (Gauss's law)**.** *Let S be a surface that encloses a charge Q, where V is the volume bounded by S (i.e. $\partial V = S$). Then the surface integral of* $\mathbf{E}$ *along the normal of S is*

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q. \tag{5.2}$$

The value of the integral is independent *of the surface S chosen.*[1]

Proof. The proof proceeds via multiple steps; in steps 1.–3. the tools are set up for 4. & 5., where the proof is completed for point charges or a charge distribution, respectively.

1. Consider a point charge q at the origin and a volume V of $0 < r_1 \le r \le r_2$, $\theta_1 \le \theta \le \theta_2$, and $\phi_1 \le \phi \le \phi_2$, which corresponds to a radial sector slice of a sphere; let $S = \partial V$. Since $\mathbf{E}$ points radially from the origin, it is perpendicular to the normal on the sides ($r_1 < r < r_2$), and parallel on the faces $r = r_1$ and $r = r_2$; hence the surface integral depends only on the faces. The field on a face is $\mathbf{E} = (1/4\pi\epsilon_0)(1/r^2)$, and the area of a face is[2] proportional to r^2, $S_r = \alpha r^2$, hence

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \oint_{S_{r_1}} + \oint_{S_{r_2}} = -E(r_1)\alpha r_1^2 + E(r_2)\alpha r_2^2 = \frac{\alpha q}{4\pi\epsilon_0}\left(-\frac{1}{r_1^2}r_1^2 + \frac{1}{r_2^2}r_2^2\right) = 0,$$

where the minus is because the surface normal is opposite the electric field.

2. Consider an infinitesimally thin surface dS, where the sides are the same as before, but a (flat) face is tilted by an infinitesimal angle $d\beta$; then the area of the face is proportional[3] to the area of the radial (flat) face: $dS_{\text{tilt}} = dS_{\text{flat}}/\cos\beta$. The normal is no longer along the electric field, and is approximately constant, taking the value at the centre, namely, $\mathbf{E}\cdot d\mathbf{a} = E(da)\cos\beta = EdS_{\text{tilt}}\cos\beta = EdS_{\text{flat}}$. Therefore the integral is unchanged.

Any surface may be made of such flat tilted segments; therefore for any surface that does not enclose a charge, the surface integral is 0.

[1] As long as it encloses the same Q.

[2] The correct value is $S_r = r^2(\cos\theta_1 - \cos\theta_2)(\phi_2 - \phi_1)$.

[3] Consider a triangle OAB, where $\overline{OA} = \overline{OB}$; let ACB be a triangle such that C is on the OB radial line and $\angle CAB = \beta$, and let BH, where H is on AC, be the height drawn up from B; then the length of AC is approximately the length of the hypotenuse AH, namely, $\overline{AC} \approx \overline{AH} = \overline{AB}/\cos\beta$.

3. Consider a single point charge Q inside V bounded by the surface S given in the theorem. Since S is bounded, there exists some radius $R > 0$ such that the sphere S_R of radius R fully encloses S (with a gap). Now consider the double surface $S' = S_R \cup S_{\text{in}}$, where the normal points inwards (into V) along $S_{\text{in}} = S$ and outwards along S_R; hence the surface does not enclose a charge, and by 1. and 2. it follows that the surface integral is zero:

$$0 = \oint_{S_R \cup S_{\text{in}}} \mathbf{E} \cdot d\mathbf{a} = \oint_{S_R} \mathbf{E} \cdot d\mathbf{a} - \oint_S \mathbf{E} \cdot d\mathbf{a}$$

hence

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \oint_{S_R} \mathbf{E} \cdot d\mathbf{a} = \frac{Q}{4\pi\epsilon_0} \frac{1}{R^2} 4\pi R^2 = \frac{1}{\epsilon_0} Q.$$

4. We now finish the proof in the case of point charges. Let there be (possibly *countably* infinitely many) charges $q_1, q_2, ...$ such that $\sum_{n=0}^{\infty} q_n = Q$ contained within the surface S. By the principle of superposition, we can separate the electric field: let $\mathbf{E}_n$ be the field from charge q_n; then, assuming that the integral distributes (is definitely true for a finite number of charges),

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \oint_S \left(\sum_n \mathbf{E}_n \right) \cdot d\mathbf{a} = \sum_n \oint_S \mathbf{E}_n \cdot d\mathbf{a} = \sum_n \frac{1}{\epsilon_0} q_n = \frac{1}{\epsilon_0} \sum_n q_n = \frac{1}{\epsilon_0} Q.$$

The theorem is thus proven for (finitely or countably infinitely many) point charges. (NB! cf. Remark 5.3 for an alternative proof.)

5. Now, we generalise to a charge distribution. Let ρ be the charge density in $\mathbb{R}^3$, and suppose it does not contain any point charges (in that case treat them separately as in part 4.). Consider all open balls of *rational* radius $\epsilon > 0$ centred at *rational* coordinates in (namely, in $\mathbb{Q}^3 \subseteq \mathbb{R}^3$); we can write any open set in $\mathbb{R}^3$ as a union of such open balls. Since there are only countably infinitely many points in $\mathbb{Q}$, the number of balls needed to cover the interior of the volume in which ρ is non-zero, is at most countably infinite! If there is charge density on the boundary, we must take 'balls' on the boundary (dimension 2) – balls intersected with the boundary. Now, we can take each $\epsilon > 0$ to be small enough at each ball such that $\rho \approx \text{const} = q$ on each ϵ-ball. Then we simply treat it as a charge $(4/3)\pi\epsilon^3 q$ at the centre of the ball, and apply part 4. By varying the coordinates and letting $\epsilon \to 0$, we will obtain convergence.

Crucially, the sum over all the charges is countable! (Uncountable sums always diverge.[4]) We can enumerate all the balls $n = 1, 2, 3, ...$ and $q_n = (4/3)\pi\epsilon^3 q_{\text{at } \epsilon\text{-ball}}$. Therefore,

$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \sum_{n=1}^{\infty} \oint_S \mathbf{E}(q_n) \cdot d\mathbf{a} = \sum_{n=1}^{\infty} \frac{q_n}{4\pi\epsilon_0} \frac{1}{\epsilon^2} 4\pi\epsilon^2 = \frac{1}{\epsilon_0} \sum_{n=1}^{\infty} q_n = \frac{1}{\epsilon_0} \int_V \rho \, d\tau = \frac{1}{\epsilon_0} Q.$$

After all this effort, the theorem finally follows also for a charge distribution. □

5.2 Remark. Steps 1.–3. are adapted from the *Feymnann lectures*, vol 2.

5.3 Remark. Step 4. may be proven in a more formal way. Again, let there be (possibly *countably* infinitely many) charges $q_1, q_2, ...$ such that $\sum_{n=0}^{\infty} q_n = Q$ contained within the surface S. Let S_R^n be the sphere of radius R centred at q_n with R large enough such that the surface is completely outside S; now for any n we have that $\oint_S \mathbf{E} \cdot d\mathbf{a} = \oint_{S_R^n} \mathbf{E} \cdot d\mathbf{a}$. Suppose q_n is isolated from other charges (there is no infinite sequence of charges that approaches it); then by picking $R_n = (1/2) \min_{m \neq n} |\mathbf{r}'_n - \mathbf{r}'_m|$, we have only one charge within the surface $S_{R_n}^n$, and thus $\oint_{S_{R_n}^n} \mathbf{E} \cdot d\mathbf{a} = q_n/\epsilon_0$.

[4] Unless only countably many terms are non-zero.

We have one edge case to deal with: suppose a sequence of isolated charges $\{q_n\}_{n\in\mathbb{N}}$ approached some other charge q_L, that is, $\mathbf{r}'_n \xrightarrow{n\to\infty} \mathbf{r}'_L$ (for any $\epsilon > 0$ there exists an N' such that $|\mathbf{r}'_n - \mathbf{r}'_L| < \epsilon$ for all $n \geq N'$). If partial sums $q_n + q_{n+1} + ... + q_{n+m}$ did not tend to 0 as $n \to \infty$, then the infinite sum $\sum_{n=0}^{\infty} q_n$ would diverge and $Q = \infty$ – a contradiction. Hence for any $\epsilon > 0$ there exists an N' such that $|q_n + ... + q_{n+m}| < \epsilon$ for all $n \geq N''$ and $m \geq 0$. Let $N = \max\{N', N''\}$. Let S_L be centred on q_L and contain only $q_N, q_{N+1}, ...$, hence the charges contribute at most ϵ to the integral over S_L, namely, $\left|\oint_{S_L} \mathbf{E}\cdot d\mathbf{a} - q_L/\epsilon_0\right| \leq \epsilon$. Letting $\epsilon > 0$ be arbitrarily small, we obtain $\oint_{S_L} \mathbf{E}\cdot d\mathbf{a} = q_L/\epsilon_0$. Therefore we can treat q_L as an isolated charge, letting $R_{n=L}$ be the radius of S_L above.

The integral over all charges $q_1, q_2, ...$ is thus

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \sum_{n=1}^{\infty} \oint_{S^n_{R_n}} \mathbf{E}(q_n)\cdot d\mathbf{a} = \sum_{n=1}^{\infty} \frac{q_n}{4\pi\epsilon_0}\frac{1}{R_n^2}4\pi R_n^2 = \frac{1}{\epsilon_0}\sum_{n=1}^{\infty} q_n = \frac{1}{\epsilon_0}Q.$$

This completes the proof.

* * *

For a charge distribution, Gauss's law may be easily rewritten into a differential form, given below.

5.4 Corollary (Gauss's law in differential form). *Let ρ be a charge density. Then*

$$\nabla\cdot\mathbf{E} = \rho/\epsilon_0. \tag{5.3}$$

Proof. Let V be an arbitrary volume. By the divergence theorem,

$$\int_V \nabla\cdot\mathbf{E}\,d\tau = \oint_{\partial V} \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}} = \int_V \frac{\rho}{\epsilon_0}\,d\tau,$$

which implies that $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$ within V; since V was arbitrary, it is true everywhere. □

We typically apply Gauss law to easily compute the electric field around a *symmetrical* charge distribution – where the magnitude and direction of $\mathbf{E}$ at every point on the surface is constant and points along the normal. Thus we can recover $\mathbf{E}$ as

$$\mathbf{E} = \frac{1}{S_{\text{surface}}}\frac{Q_{\text{enc}}}{\epsilon_0}\hat{\mathbf{n}}, \tag{5.4}$$

where $\hat{\mathbf{n}}$ points along the surface normal.

§6. Curl and potential of the electric field

We compute the curl of the electric field given by a charge density ρ as follows

$$\nabla\times\mathbf{E} = \nabla\times\left(\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}\,d\tau'\right) = \frac{1}{4\pi\epsilon_0}\int_V \nabla\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\mathbf{r}')\,d\tau' = 0 \tag{6.1}$$

since the curl of $\hat{\mathscr{r}}/\mathscr{r}^2$ is zero. This gives the third Maxwell's equation for static electric fields: $\nabla\times\mathbf{E} = 0$.

By Helmholtz's theorem, $\mathbf{E} = -\nabla U + \nabla\times\mathbf{W}$ for some potential U and vector field $\mathbf{W}$, where

$$U(\mathbf{r}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla\cdot\mathbf{E}(\mathbf{r}')}{\mathscr{r}}d\tau' \neq 0 \qquad \text{and} \qquad \mathbf{W}(\mathbf{r}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla\times\mathbf{E}(\mathbf{r}')}{\mathscr{r}}d\tau' = 0, \tag{6.2}$$

thus $\mathbf{E} = -\nabla V$ for some function $V : \mathbb{R}^3 \to \mathbb{R}$ called the *(electric) potential* (we use V instead of U). Since only the derivatives of V influence $\mathbf{E}$, it follows that we may offset V

by any constant without changing $\mathbf{E}$ – usually we let $V = 0$ at infinity. Then $V(\mathbf{r})$ is the work required to bring a unit charge to $\mathbf{r}$ from infinity.

Now, we compute

$$\frac{\rho}{\epsilon_0} = \nabla \cdot \mathbf{E} = -\nabla \cdot (\nabla V) = -\nabla^2 V,$$

giving *Poisson's* equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}. \tag{6.3}$$

Indeed, we may compute V directly from (6.2), obtaining by Gauss's law

$$V(\mathbf{r}) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\nabla \cdot \mathbf{E}(\mathbf{r}')}{\mathscr{r}} d\tau' = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')/\epsilon_0}{\mathscr{r}} d\tau' = \frac{1}{4\pi\epsilon_0}\int_{\mathbb{R}^3} \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'. \tag{6.4}$$

We may recover V in the form of a line integral as follows. Let $\mathcal{O} \in \mathbb{R}^3$ be a reference point, and let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ be two points. Then the line integral of the electric field due gives

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\boldsymbol{\ell} = -[V(\mathbf{b}) - V(\mathbf{a})],$$

where we set $V(\mathcal{O}) = 0$ at the reference $\mathcal{O}$, which implies

$$-V(\mathbf{r}) = -[V(\mathbf{r}) - 0] = -[V(\mathbf{r}) - V(\mathcal{O})] = -\int_{\mathcal{O}}^{\mathbf{r}} \nabla V \cdot d\boldsymbol{\ell} = \int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell},$$

hence

$$V(\mathbf{r}) = -\int_{\mathcal{O}}^{\mathbf{r}} \mathbf{E} \cdot d\boldsymbol{\ell}. \tag{6.5}$$

Note that (6.5) is the definition of potential chosen in the book – this shows that the argument follows both ways, hence all definitions are equivalent. The principle of superposition extends to V.