mathnet.ru/rus/conf1591

Deformations of Poisson brackets according to Kontsevich

Finite graphs w/ structure ← 3rd natural example of a Lie Algebra

§1 — Kontsevich's graph complex.

AVK., IUM., 13/May/2019

- M.K.: ~~1998~~ 1993-4, 6-7, 2017 (Bourbaki) congress ['93-'94 congress Ascona! '96] "formality conjecture"
pioneering work — deformation quantization almost understood → understood
- Willwacher, ETH Zürich; Rodgers, Dolgushev, Zivkovich; 2010-17 1009.1654;
Annales, advances   Christine Jost [1211.4230, 1012.2467]
- AVK + company.

Differential Lie Algebra (graded) on graphs (DGLA Graphs)

NB] many definitions ← Willwacher, Merkulov

γ — finite ($\# V_\gamma < \infty$), "connected" (usually, but not obligatory),
"without leaves" (skip 1 edge → graph splits into 2 parts)
"strongly connected" (skip 1 vertex → graph splits)

"$N(v \in Vert) \geqslant 3$"
number of neighbours is at least 3 ← ? why =2,1 are irrelevant cases

- no multiple edges 
- no tadpoles 
- non-oriented edges

{for today: $\mathbb{R}$} → $\mathbb{K}$; satisfies assumption above

Vector space of graphs $\gamma = \sum_a^{\ell} c_a \gamma_a$, $c_a \in \mathbb{K}$

Wedge order:

$$E(\gamma_a) = \bigwedge_{e_i \in Edge(\gamma_a)} e_i = I \wedge II \wedge \cdots \wedge IV$$

labeling edges ordered

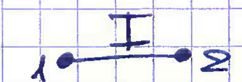
Space is dim $= \infty$, but any slice of fixed #Vert is finite-dim.

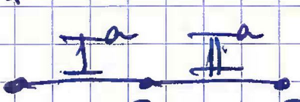
§1 Space $\left(Gra_{\#Vert \geqslant 1 =: n}^{\wedge_i e_i} \right) / S_n \;/\cong$ ← order; unlabeled vertices

NB] zero is very large ← graph equal to itself

Relation: $\cong$: $\sigma: \gamma_a \overset{top}{\cong} \gamma_b$ ↦ σ_V, $\sigma_E: Edge(\gamma_a) \rightleftarrows Edge(\gamma_b)$

$\gamma_b = (-1)^{\sigma_E} \gamma_a$
if even, then equal,
if odd, then opposite

Ex: 1. $\mathcal{E} =$  (already has leaves)

2. $\gamma_a =$ 

$I^a \wedge II^a = E(\gamma_a)$

$= \gamma_b$, $E(\gamma_b) = I^b \wedge II^b$

graph γ has a symmetry: ⇆
graphs $\gamma_a \cong \gamma_b$

{Zero graph}

Iso $\sigma: I^a \cong II^b$, $II^a \cong I^b$

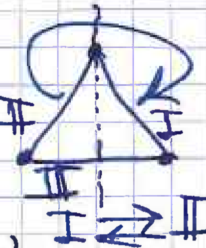
We see that by slt bijection on vertices $\gamma_a \overset{equal}{\cong} \gamma_b$

$I^a \wedge II^a = II^b \wedge I^b$
$= -II^a \wedge I^a$ (=)

① pedagogical arXiv ↓
1710.00658 ✓
1811.10638 ✓
[Type "a"]
② general algebra
1608.01710 ✓
1602.09036 ✓
infinitesimal deforms
③
1811.07878
1904.13293

Additional:
1210.0726
"B-field fields" Schouten bracket
1702.00681
⋆ product up to $\hbar^4$
1705.01777
⋆ in field theory

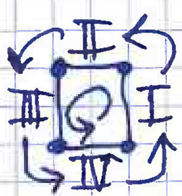
Ex 3. $\boxed{2}$



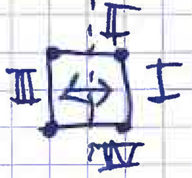
to show zero graph, must invert a symmetry (automorphism) w/ odd parity permutation of edges } "zero if symmetry has odd parity" rule of thumb

II ↺, I ⇄ III ⇒ $\Delta = 0 \in \mathrm{Gra}$

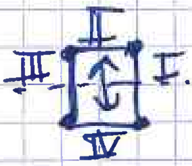
4.



$(-1)^\sigma = -1$,

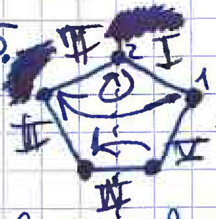


$(-1)^\sigma = -1$



$(-1)^\sigma = -1$

5.

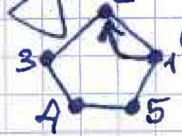


$\overset{?}{=} 0$

automorphism preserves symmetry

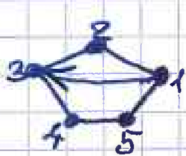
by symmetry $1 \to 2$ or 3 (2 cases)

$1 \to 2$:

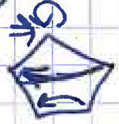


⇒ $\underbrace{5 \to 1}$ or 3

$1 \to 3$:

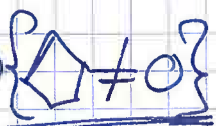


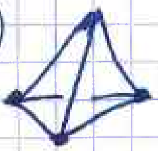
⇒ $5 \to \underbrace{2}$ or 4 ⇒

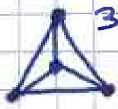


$4 \to 1$
$3 \to 5$

2 ?

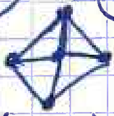
⇒ {  $\neq 0$ }

?  $\neq 0$



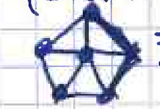
3-wheel

? 2k-gon wheel



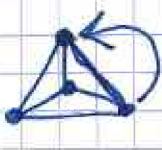
$\overset{?}{=} 0$

(2l+1)-gon wheel

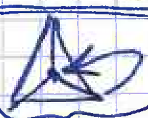


$\overset{?}{\neq} 0$

6.



or



→ rotation does not reduce to other cases

⇒  $(-1)^\sigma = (-1)^4 = ①$ even

spatially does nothing

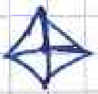
even as comp of 2 even

has $(-1)^\sigma = 1$ as product of 2 traps

How to generalise to $(2l+1)$–?

rotation does not work; interchange has even legs ⇓ comps of even crossed ∎.

7.

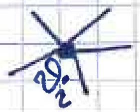
gives $\sigma = (1\,2)(2\,3)(3\,4)$ ⇒  $= 0$ in Gra

$(-1)^\sigma = \underline{-1}$

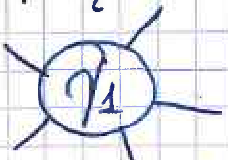
immediately generalises to $2l = n$.

DEF Insert $\circ_i : \gamma_1 \longrightarrow \begin{pmatrix}\text{vertex } v_i \\ \text{in graph } \gamma_2\end{pmatrix}$ extended linearly

$$\gamma_1 \circ_i \gamma_2 = \boxed{\gamma_1} \longrightarrow v_i \in \mathrm{Vert}(\gamma_2)$$

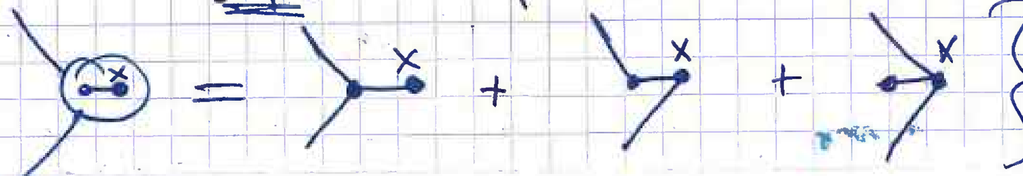


{blow up}



} formal sum of graphs (edge to vertices independently w/o edges)

Ex:



Postulate $E(\gamma_1 \vec{\circ}_i \gamma_2) = E(\gamma_1) \wedge E(\gamma_2)$

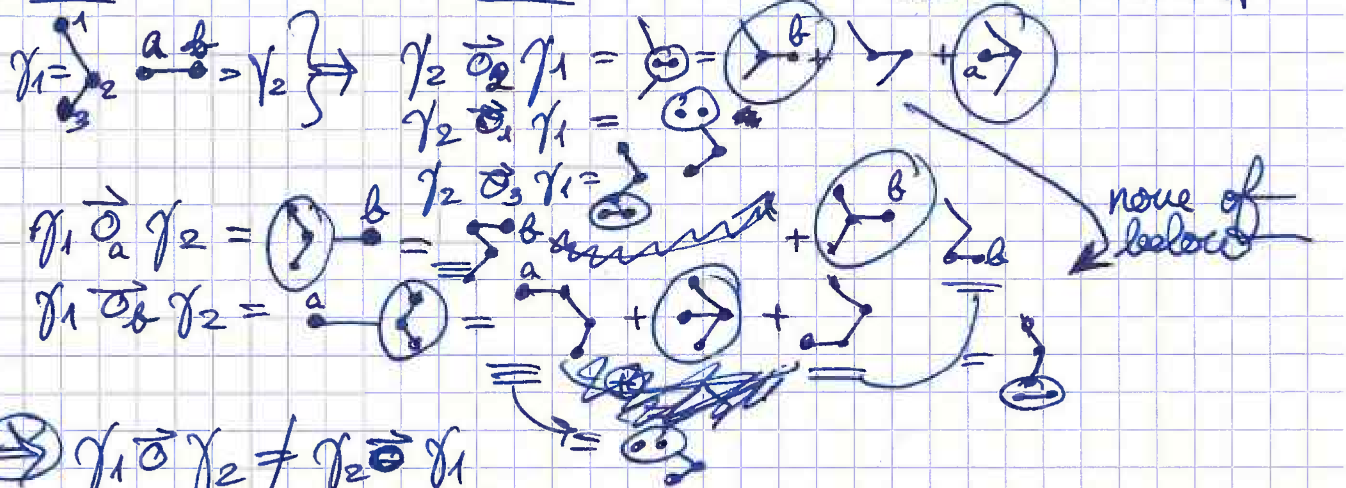
$\left\{ \begin{array}{l} \#V(\gamma_1) + \#V(\gamma_2) - 1 \\ \#E(\gamma_1) + \#E(\gamma_2) \end{array} \right\}$ ← resulting values

Some literature: B

$\gamma_1 \vec{\circ}_i \gamma_2$ vs $\gamma_1 \overleftarrow{\circ}_i \gamma_2$

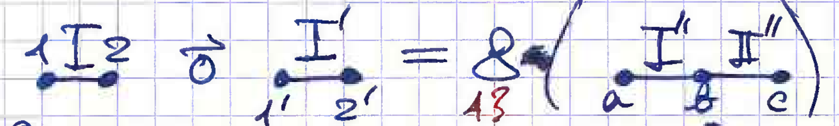
DEF: $\gamma_1 \vec{\circ}_\bullet \gamma_2 = \sum_{v_i \in \mathrm{Vert}(\gamma_2)}^{\ell} \gamma_1 \vec{\circ}_i \gamma_2$

NB! not commutative, not associative ? (Counter)examples

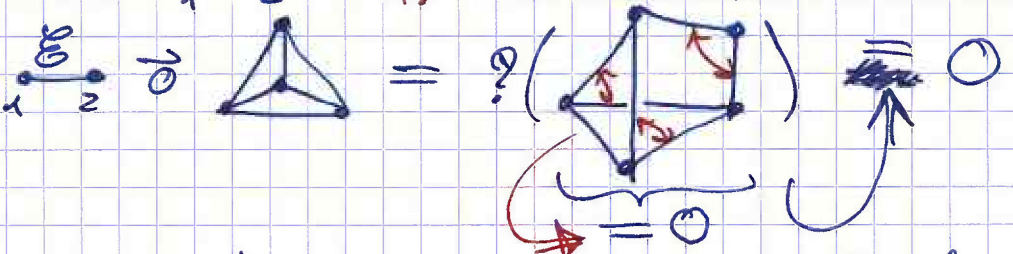


$\Rightarrow \gamma_1 \vec{\circ} \gamma_2 \neq \gamma_2 \vec{\circ} \gamma_1$

Ex:



Ex: ?



③ Lie $[\cdot,\cdot]$ on Gra. ← everything is ext. linearly; no statements about connectivity

$[\gamma_a, \gamma_b] = \gamma_a \vec{\circ} \gamma_b - (-1)^{\#E(\gamma_a)\#E(\gamma_b)} \gamma_b \vec{\circ} \gamma_a$

Bi-linear by linear ext; graded skew-symmetric by construction;

not ~~[illegible]~~ skew (−1)orders – skew ← similar to exterior algebra (tensors)

- Satisfies Jacobi identity:

$[a, [b,c]] - (-1)^{|a||b|} [b, [a,c]] = [[a,b],c]$

"γ_a"; swapping $E(\gamma_b) \wedge E(\gamma_a)$; #edges; [1811.10638] Thm; "really" = "not modulo 0"

"how to understand": "commutator of adjoint actions is adjoint action of commutator"

1. fix c
2. $[a,[b,c]]$ – ( ) $[b,[a,c]]$ = $[[a,b],c]$

action by b then by a; by a then b; by commutator

often in noncommutative settings is easiest/only way of proving Jacobi

Pf(sketch): • look at $c \Rightarrow$ "$=0$", • LHS (Jacobiator) − RHS is skew ← Koszul sign w.r.to S_3 on "a, b, c"

$\boxed{4}$ Koszul sign: $\epsilon(1 \rightleftarrows 2) = - (-)^{|1|\cdot|2|}$

↑ due to transposition of 2 objects

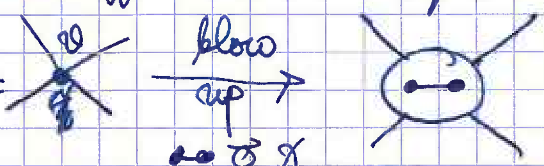
§3a [1811.10638] Check that everything is well-defined!
[1710.00658]

(?) $[\gamma, \text{zero}] \overset{\text{Th}}{=} 0\left(\text{some nonzero graphs}\right) + \left(\text{zero graphs}\right) = 0$ } group actions, orbits

(∀)

§3b Differential $d_r(\gamma) = [\overset{\varepsilon}{\bullet\!-\!\bullet}, \gamma]$ insertion of an edge

Ex: $\gamma =$



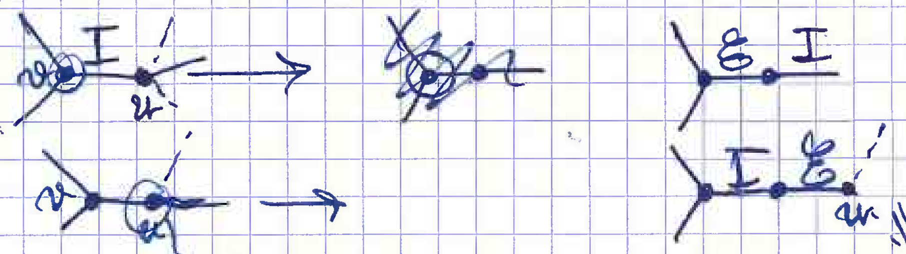
$\bullet\!-\!\bullet \circ \gamma$

$\gamma \circ \overset{\varepsilon}{\bullet\!-\!\bullet} = \gamma\overset{\varepsilon}{-\bullet} + \overset{\varepsilon}{\bullet-}\gamma$

(?) All new leaves cancel out.

Rem: No leaves in $\gamma \Longrightarrow$ no leaves in $d_r(\gamma)$

(?) [handshake lemma]: If $\forall v \in \text{Vert}(\gamma) \mid \# N(v) \geqslant 3$, then $\{\forall w \in \text{Vert}(d_r(\gamma)) \Rightarrow \# N(w) \geqslant 3\}$.

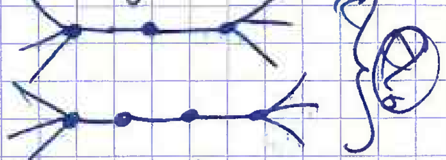


} graphs differ only by transp. of 2 edges $\Longrightarrow$ are ⊖ opposite

$\Longrightarrow$ cancel out

(?) Find out what happens if vertex $v \mid N(v) = 3$ is blown up by $d_r = [\bullet\!-\!\bullet, \cdot\,]$ ← cancel out

(?) Snake of even vs odd length



} (?)

We pay attention only to vertices of valency at least 4.

$d_r(\text{zero}) = \text{zero}$. $\Longrightarrow$ differential well-defined on the quotient

§3c (?) $\boxed{d_r^2 = 0}$ ← really 0(graph) "vanishes not just in Gra"

stronger

Rem: (?) not via Jacobi identity

$\bullet\!-\!\bullet\!-\!\bullet = 0 \in \text{Gra}$

$[\overset{\varepsilon}{\bullet\!-\!\bullet}, [\overset{\varepsilon'}{\bullet\!-\!\bullet}, \gamma]] - \underbrace{(-1)^{1\cdot 1}}_{=+} [\overset{\varepsilon'}{\bullet\!-\!\bullet}, [\overset{\varepsilon}{\bullet\!-\!\bullet}, \gamma]] = [\overbrace{[\bullet\!-\!\bullet, \bullet\!-\!\bullet]}, \gamma] = \underset{\neq 0}{\text{const}}(\bullet\!-\!\bullet\!-\!\bullet) = \overset{\text{Gra}}{\overset{\cup|}{0}}$

$\Longrightarrow 2\, d_r^2(\gamma) = \underset{\neq 0}{\text{const}}(\bullet\!-\!\bullet\!-\!\bullet)$ "vanishes in Gra"

weaker

[Time 1:47 min]

Differential graded Lie Algebra (dgLa)

Challenges: 21st century so far ~ ~~[illegible]~~ Teichmüller [Willwacher]

cocycle – element of ker d; coboundary – of im d; cohomology: ker d / im d

(3.4) dr-cocycles. $dr = [•—•, \cdot]$

$\gamma = \mathcal{E} =$ •—• ∈ ker dr

(?) Exact?: $dr(\bullet) = [•—•, \bullet]$ = •—• ∘ • − $(-1)^{1\cdot 0}$ • ∘ •—•

= ⦿—• − { ⊙—• + •—⊙ }

= $\overset{\mathcal{E}}{•—•}$ − $\overset{\mathcal{E}}{•—•}$ − $\overset{\mathcal{E}}{•—•}$

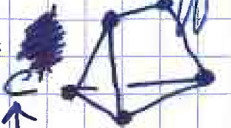
= − •—• ⇒ yes, ~~[illegible]~~ •—• is exact

⇒ •—• is a trivial cocycle

Ex [Ascona '96]

$\gamma_3 =$ 

when we blow up one vertex, all leaves in differential cancel

$dr(\gamma_3) =$ C  = 0

$C = 4 \cdot 2 \cdot 2 \cdot ?$

#vertices | 2 ← new old or old new | choice of face adjacent to old new edge

(?) trivial, then obtained by blowing up one ~~[illegible]~~ vertex into edge — which?

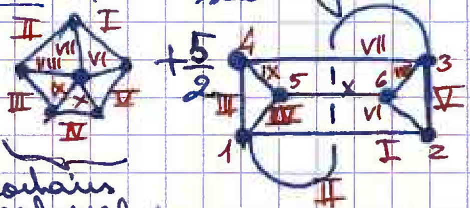
(by sym. new edge ⇒ before edge was contracted

contract ⇒ double edge ⊘ (=) zero graph

reason why we restrict to simple graphs morally

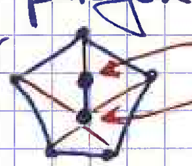
{any graph w/ 3 vertices = 0}.

generalises to all wheels of odd # spikes

$\gamma_5 =$  $+ \frac{5}{2}$ [Kont., Willw.]

contains nontrivial ⇒ γ_5 nontrivial

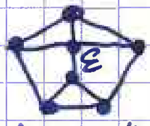
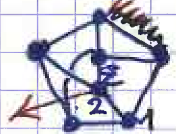
"pentagon wheel cocycle"



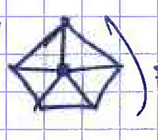
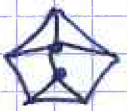
conn. to 1 → vertex of valency 2

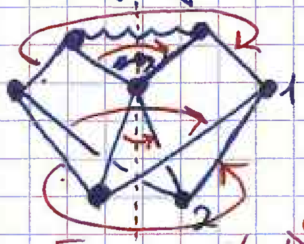
conn. to 4 ⇒ cancels [why?]

⇒ must be conn. as 3|2

10  + or  {how many?} 5 edges ⤳ 2 go to 1 vertex of $\mathcal{E}$ ⇒ 10

"human" "monkey"

⇒ $dr\Big($  $\Big) = 10$ 



5 pairs $(-1)^{\sigma} = (-1)^5 = -1$

[1410.00658] explicit labeling

#V = 5 ⇒ all trivial

#V = 6 ⇒ only this

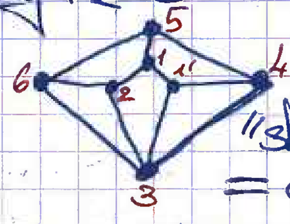
Examples: $(\#V, \#E) = (n, 2n-2)$

blow up 1 of prism graph: Claim: 1 ⟶ $\overset{1}{•}\underset{\mathcal{E}}{—}\overset{1'}{•}$; 1 is adjacent to 1′, 2, 3 / 1′, 2, 4 / 1′, 2, 5 (?) always to 1′, 2

(?) first: which is −2 (human) · 2 "symmetry 1 ⇄ 3"

which is ∝ (monkey) = 0 ∈ Gra

which is new graph "stone"



"stone" = 0 ∈ Gra

⇒ $dr(\gamma_5) = 10$ "human" $+ \frac{5}{2} \cdot (-4)$ "human" $+ 0 = 0$ "human" ; $\gamma_5 \in$ ker dr nontrivial

Thm [Dolgushev – Rodgers – Willwacher, 1211.4330, Ann. Math, 2015]:

$\forall \ell \in \mathbb{N}_{\geq 1}\ \exists \gamma \in \mathrm{Gra} \mid \mathrm{d}r(\gamma)=0$, $\underbrace{\gamma \notin \mathrm{im}\,\mathrm{d}r}_{\text{nontrivial}}$,

$\gamma = 1\{(2\ell+1)\text{-wheel}\} + \underbrace{\{\cdots\}}_{\text{open problem}}$

each wheel of odd spikes is a marker for cocycles

unique cocycles

Ex: $\gamma_3, \gamma_5, \gamma_7, [\gamma_3, \gamma_5], \gamma_9$

NB! [dr-cocycle, dr-cocycle] is a dr-cocycle

Prop Free Lie algebra generated by $\gamma_{2\ell+1}$

Thm [Willwacher, Invent., 2015]. Cohomology group $H^0(\mathrm{Gra}, \mathrm{d}r) \xrightarrow[\text{isomorphic}]{\simeq}$

$H^0(\mathrm{Gra}, \mathrm{d}r) \simeq \underbrace{\mathfrak{grt}}$ Lie algebra of (GRT) by [Drinfeld, 1990]

two very distant parts of mathematics related via isomorphism

$\mathfrak{grt} \bullet \longleftrightarrow \bullet\, \mathrm{Gra} \xrightarrow[O\vec{\Gamma}]{} \left\{ \begin{array}{c} \text{symmetries} \\ \dot{P} = \underbrace{Q}_{\text{polynomial}}(P) \end{array} \right\}$

"infinitesimal deformation"

how many symmetries are not in im(Gra) ? open problem

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§L2 – Symmetries of Poisson brackets $\mathrm{sym}\{\cdot,\cdot\}_P$

Lecture material [1608.01710] [1712.05259]

§1 Poisson brackets on smooth functions on $N^{r<\infty}$

$C^\infty(N^{r<\infty}) \leftarrow$ affine

only shifts & rotations allowed

e.g. circle w/ angle θ

DEF: bilinear, skew (anti-symmetric), bi-derivation (product rule on each argument), Jacobi identity; denoted $\{\cdot,\cdot\}_P$

$$\sum_{\circlearrowleft} \{\{a,b\}_P, c\}_P = 0$$

(sum over cyclic permutations)

Ex: $(\mathbb{R}^3)$. $\{f,g\}_{\det} := \det \underbrace{\left(\frac{\mathcal{D}(a,f,g)}{\mathcal{D}(x^1,x^2,x^3)}\right)}_{\text{Jacobian}} \leftarrow$? Jacobi identity

$a \in C^\infty(\mathbb{R}^3)$ some fixed function

for $\mathbb{R}^{n\geq 3}$, $\det\left(\frac{\mathcal{D}(a^1,\ldots,a^{n-2},f,g)}{\mathcal{D}(x^1,\ldots,x^n)}\right)$

$\forall f(x^1, x^2, x^3)$ let $\{f, g\}_{3D} = f \cdot \{f, g\}_{\det(\mathbb{R}^3)}$ $\Big\}$ $\mathcal{P} = \dfrac{da}{d\mathrm{vol}(x)}$
distribution

Classification of all Poisson brackets on $\mathbb{R}^3$
open problem.

$$\{f,g\}_{\mathcal{P}} = (f)\frac{\overleftarrow{\partial}}{\partial x^i}\bigg|_{\underline{\underline{x}}} \overset{\uparrow}{\cdot} \mathcal{P}^{ij}(\underline{\underline{x}}) \cdot \frac{\overrightarrow{\partial}}{\partial x^j}\bigg|_{\underline{\underline{x}}}(g)$$

local coordinates $\underbrace{x^1, \ldots, x^r}_{\text{even}}$ parity

denote coupling of vectors and bivectors $\leftarrow \underline{\underline{?}}$

$$\mathcal{P} = \|\underbrace{\mathcal{P}^{ij}}_{\text{coeff of bivector}}(\underline{\underline{x}})\|$$

Poisson bi-vector

Generalisation: $\dfrac{da_1 \wedge \cdots \wedge da_{n-2}}{d\mathrm{vol}(x)}$

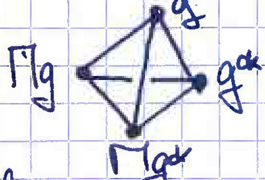
Generalisation: Vinogradov brothers
1997
(N, k, r)-family
$\Rightarrow$ Lie

Nambu ← one extreme case
Schlessinger / Stasheff ← opposite

Tangent bundle over manifold TN^r; dual T^*N^r (cotangent bundle)
parity-reversed fibors ΠT^*N^r local coordinates $\underbrace{\xi_1, \ldots, \xi_r}_{\text{odd parity}}$

$$\mathcal{P} = \frac{1}{2}\mathcal{P}^{ij}(\underline{\underline{x}})\,\xi_i\,\xi_j$$

$$\xi_i(x^j) = \delta_i^j$$



$$\{f,g\}_{\mathcal{P}} = (f)\frac{\overleftarrow{\partial}}{\partial x^i}\bigg|_{x} \cdot \frac{\overrightarrow{\partial}}{\partial \xi_i}(\mathcal{P})\frac{\overleftarrow{\partial}}{\partial \xi_j} \cdot \frac{\overrightarrow{\partial}}{\partial x^j}(g)\bigg|_{x}$$

Andrei Krutov, Kiselev
generalisation to field theories (homol. realisation)
⊕ integrability

[1312.1262] BV formalism

Generalisation to variational
[1705.01777] to field (Poisson) theories

Dots become couplings ← explicit

Vector fields on manifolds; commutator of 2 vector fields $[\mathcal{X}, \mathcal{Y}]$

Schouten bracket $[\mathcal{X}, \mathcal{Y}] \rightsquigarrow [\![\mathcal{P}, \mathcal{Q}]\!]$
1935, Groningen vectors multi-vectors

$$[\mathcal{X}, \mathcal{Y}](f) = \mathcal{X}(\mathcal{Y}(f)) - \mathcal{Y}(\mathcal{X}(f))$$
$$f \in C^\infty(N^r)$$

$$\mathcal{P} = \overset{\uparrow}{\mathcal{X}_1} \wedge \mathcal{X}_2 \wedge \cdots \wedge \mathcal{X}_k$$
vector field realised thru ξ_i

$$[\![\mathcal{X}, \mathcal{Y} \wedge \mathcal{Z}]\!] \overset{\uparrow}{=} [\![\mathcal{X}, \mathcal{Y}]\!] \wedge \mathcal{Z} + (-1)^{(|\mathcal{X}|-1)|\mathcal{Y}|}\, \mathcal{Y} \wedge [\![\mathcal{X}, \mathcal{Z}]\!]$$

(Graded) Leibniz rule

for vectors Schouten coincides w/ commutator.

$$[\![\mathcal{X}, \mathcal{Y}]\!] = -(-1)^{(|\mathcal{X}|-1)(|\mathcal{Y}|-1)}[\![\mathcal{Y}, \mathcal{X}]\!]$$

for 1-vectors $[\![\mathcal{X}, \mathcal{X}]\!] = [\mathcal{X}, \mathcal{X}] = 0$

2-vectors $[\![\mathcal{P}, \mathcal{P}]\!] = \underbrace{-(-1)^{(2-1)(2-1)}}_{=\oplus}[\![\mathcal{P}, \mathcal{P}]\!]$ $\underline{\underline{?!!}}$ 0 Tautology, does not vanish

$$\llbracket \mathcal{P}, \mathcal{Q} \rrbracket = (\mathcal{P}) \frac{\overleftarrow{\partial}}{\partial \xi_i} \bullet \frac{\overrightarrow{\partial}}{\partial x^i} (\mathcal{Q}) - (\mathcal{P}) \frac{\overleftarrow{\partial}}{\partial x^i} \bullet \frac{\overrightarrow{\partial}}{\partial \xi_i} (\mathcal{Q})$$

mistake in sign in some papers

quantisation of gauge systems in BV formalism "antibracket"
[1210.0726]
M.K.'1992

symplectic Poisson bracket

footnoes multiplication ; in field theories : "variational derivative"

Ⓟ P ~~Poisson~~ Poisson $\Longleftrightarrow$ $\boxed{\llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0}$ ← Jacobi for Poisson

trivector $\forall$ $\llbracket \mathcal{P}, \mathcal{P} \rrbracket (f, g, h) = 0$

$\mathcal{P} = \frac{1}{2} \mathcal{P}^{ij}(x) \xi_i \xi_j$

$$\llbracket \mathcal{P}, \mathcal{Q} \rrbracket = -(-1)^{(|\mathcal{P}|-1)(|\mathcal{Q}|-1)} \llbracket \mathcal{Q}, \mathcal{P} \rrbracket \Big\} \text{shifted-graded skew}$$

BV formalism — current formalism key "not coherently inverse to pt"

Jacobi: for Schouten

$$\llbracket \mathcal{P}, \llbracket \mathcal{Q}, \mathcal{R} \rrbracket \rrbracket - (-1)^{(|\mathcal{P}|-1)(|\mathcal{Q}|-1)} \llbracket \mathcal{Q}, \llbracket \mathcal{P}, \mathcal{R} \rrbracket \rrbracket = \llbracket \llbracket \mathcal{P}, \mathcal{Q} \rrbracket, \mathcal{R} \rrbracket$$

Rem: 1. Claim (tomorrow) : $[\pi_g, \pi_g]_{RN} = 0$ ← Richardson – Nijenhuis

2. Assume $\llbracket \mathcal{P}, \mathcal{P} \rrbracket = 0$
(a bivector $\mathcal{P}$ is Poisson)
let $\mathcal{Q} = \mathcal{P}$ $\leadsto$ $\llbracket \mathcal{P}, \llbracket \mathcal{P}, \mathcal{R} \rrbracket \rrbracket - \underbrace{(-1)^{(2-1)(2-1)}}_{=1} \llbracket \mathcal{P}, \llbracket \mathcal{P}, \mathcal{R} \rrbracket \rrbracket =$

$$= \llbracket \underbrace{\llbracket \mathcal{P}, \mathcal{P} \rrbracket}_{=0}, \mathcal{R} \rrbracket$$

$$2 \llbracket \mathcal{P}, \llbracket \mathcal{P}, \mathcal{R} \rrbracket \rrbracket = 0$$

but we have $\llbracket \mathcal{P}, \llbracket \mathcal{P}, \cdot \rrbracket \rrbracket = \partial_{\mathcal{P}}^2$, $\partial_{\mathcal{P}} = \llbracket \mathcal{P}, \cdot \rrbracket$

$\Rightarrow$ $\boxed{\partial_{\mathcal{P}}^2 = 0}$ Adjoint action by Schouten using Poisson $\mathcal{P}$ a ~~bivector~~ differential

"Poisson differential"

Cohomology groups (Poisson):

0-vector $H^0_{\mathcal{P}}(\cdot)$ ← Casimirs ← functions which Poisson-commute w/ $\forall$ (integrable systems)

1-vector $H^1_{\mathcal{P}}(\cdot)$ ←

2-vector $H^2_{\mathcal{P}}(\cdot)$ ← infinitesimal deformations of Poisson bi-vector, which are not of form $\mathcal{Q} = \llbracket \mathcal{P}, \overset{\uparrow}{\mathcal{X}} \rrbracket$ 1-vector

3-vector $H^3_{\mathcal{P}}(\cdot)$ ← Ⓟ obstruction to integration of infinitesimal deformations to finite.
(our lectures)

Ⓟ Deformation $\mathcal{P} \mapsto \mathcal{P} + \varepsilon \underbrace{\llbracket \mathcal{P}, \mathcal{X} \rrbracket}_{2\text{-vector}} + \bar{o}(\varepsilon)$ ← $H^2_{\mathcal{P}}(\cdot)$

, [trivial] Poisson cocycle

has a very nice geometry

§1C

9

For a bi-vector to be Poisson (dim < ∞) $[\![P,P]\!]=0$ (master equation)

Deform $P \mapsto P + \varepsilon Q + \bar{o}(\varepsilon)$: remains Poisson in linear approx:

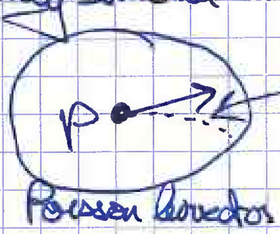
$$[\![P+\varepsilon Q+\bar{o}(\varepsilon), P+\varepsilon Q+\bar{o}(\varepsilon)]\!] = \bar{o}(\varepsilon)$$

expanding,

$$[\![P+\varepsilon Q+\bar{o}(\varepsilon), P+\varepsilon Q+\bar{o}(\varepsilon)]\!] = \underbrace{[\![P,P]\!]}_{=0} + \varepsilon\{\underbrace{[\![P,Q]\!]+[\![Q,P]\!]}_{\substack{= 2[\![P,Q]\!] \text{ (for bi-v.)}\\ \text{must be } 0}}\} + \bar{o}(\varepsilon)$$

$$\Leftrightarrow \boxed{[\![P,Q]\!]=0} = \partial_P(Q)$$

Only some bi-vectors deformable or most?



: stays Poisson or almost?

Can this be done universally? ← isolated / families

We are searching for ∂_P cocycles: $\partial_P = 0$

Q depends on P universally (all N and all P on N) : $\partial_P(Q(P)) = 0$

$[\![P, Q(P)]\!]$ ← contained $[\![P,P]\!]=0$ in disguise

Claim (?) [Ascona '96] $(\forall N_{aff}^{d<\infty}, \forall P)$ ∃ at least countably many direction Q, which span at least dim ≥ countable: deform remains infinitesimally Poisson

⊕ [Bourbaki '2017, 1608.01710]

"All Poisson structures are deformable"

Done via Kontsevich graph calculus. {Oriented graphs}

---

§2 Oriented graphs:

[Cattaneo – Felder (2000) CMP] relation to Feynman

DEF: $\sum_{i=1}^{r} \overleftarrow{\frac{\partial}{\partial \xi_i}} \bullet \overrightarrow{\frac{\partial}{\partial x^i}} \overset{\text{def}}{=} \bullet \xrightarrow{i}$

object $\partial/\partial\xi_i$ — object which is differentiated $\partial/\partial x^i$

① $P = \bullet = \frac{1}{2} P^{ij}(x)\, \xi_i \xi_j$

Oriented graph $\overset{\text{def}}{=} \sum_{\{i\}} \prod_{\{\text{vertex}\}}$

we will place multivectors expressed as $\bullet \xrightarrow{i}$ {we use N_{aff} here} {to make coord-free} ⇔ Jacobian is constant

Ex: ② or sinks of orgraph

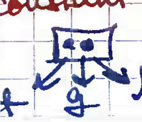
$\{f,g\}_P =$ "Left" "Right" specified ordering

Two sinks f, g & Poisson structure

$$= (f)\overleftarrow{\frac{\partial}{\partial x^i}}\overrightarrow{\frac{\partial}{\partial \xi_i}}(P)\overleftarrow{\frac{\partial}{\partial \xi_j}}\overrightarrow{\frac{\partial}{\partial x^j}}(g)$$

sink — no arrows come out

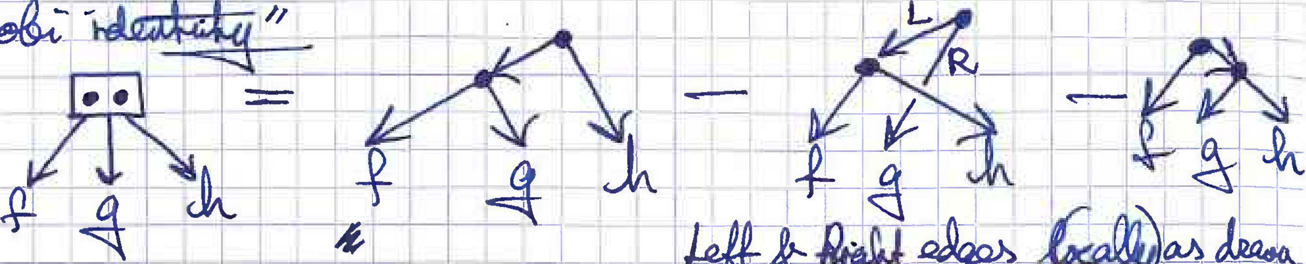
③ (or Jacobi identity) ⟶



$= \mathrm{Jac}(P)$

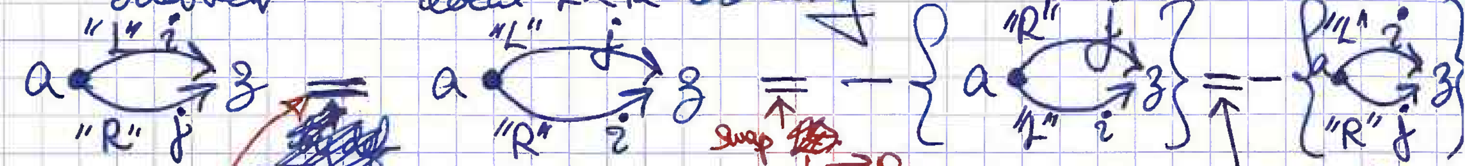
$\boxed{10}$

Ex: "Jacobi identity"



Left & Right edges (locally) as drawn

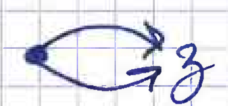
NB! About Kontsevich ordered graphs: ⇒ ∃ zero graphs
unoriented — global order
oriented — local L<R ordering



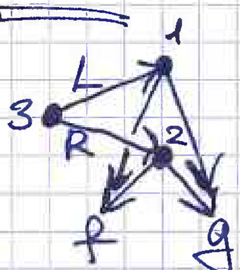
i, j ← summation indices ⟺ dummy variables

identically equal L<R

⇒



is a zero graph



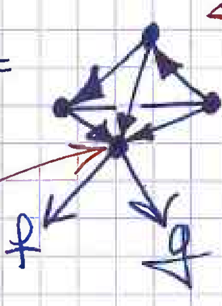
= 0 by expanding the composition

= 0 because P satisfies Jacobi

Solutions Q(P): Yes, exist. Yes, universal. (skew-symmetrised by construction)

← cocycle

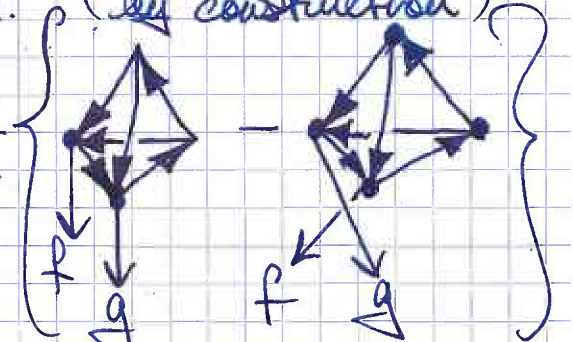
$Q_{1:\frac{6}{2}}(P) =$



← ④ Poisson structures (in each • one)
quartic in P

skew-sym ⇒ graph is bi-vector

$+ \frac{6}{2}$



$\frac{1}{2}$ because sum over 2 graphs

$n=4, \#E = 2\cdot 4 - 2 + \underset{\text{edges to sinks}}{\underline{2}}$
$= 2n$

⇒ graph is a graph built of forks ∧ ← every fork is a coeff of Poisson bi-vector

How to verify $Q_{1:6/2}(P)$ is universal (cocycle)

Schouten bracket:

$(\pm)[\![P,Q]\!] =$

dep. on how we count sinks

show: equal to zero graph.

§3 Mechanism $[\![P, Q(P)]\!] = \Diamond(P, \mathrm{Jac}(P))$ ‖ Leibniz graphs [1602.09036]

"acts on argument"

encodes a diff. polynomial expression

⇒ must obtain 0 by using it

differential order

1:1:1

2:1:1 1:2:1 1:1:2

homogeneity orders

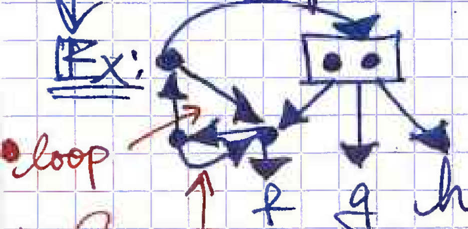
vanish ⇔ all homog comps vanish indep.

⇔ each comp is = 0

= 0 = 0 = 0 = 0

Leibniz graph: vertices contain (1) copy of P, (2) ~~sorted~~ sinks (3) Jacobiator as one vertex

Ex:



5 copies of P, each i_j

⇒ 10 summation indices

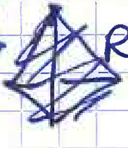
"Typical graph"

- 2-cycle
- no tadpoles
- ⩾1 one arrow on Jac
- Jac does not stand ∀ arguments
- operator not 3-vector (not skew) can be skewed: $[f \; g] \; h$

---

what kind of graphs should we expect in $\Diamond(P, \mathrm{Jac}(P))$ Leibniz graphs

$\gamma_3 \xrightarrow{RHS}$

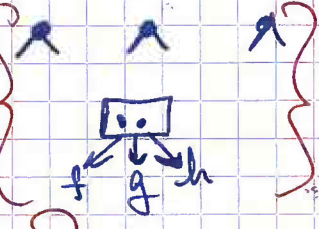


RHS: 5 internal vertices

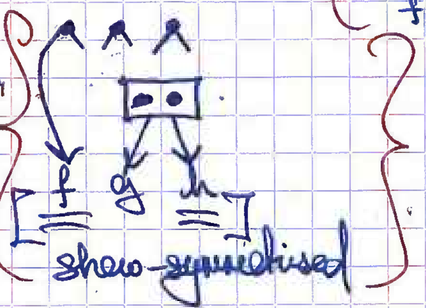
must obtain a trivector & every sink is target of one arrow

\# graphs of each type up to relabelling $\Sigma = 1132$

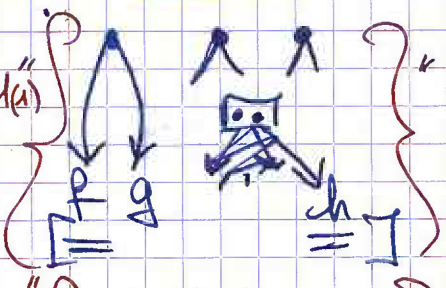
"№3"



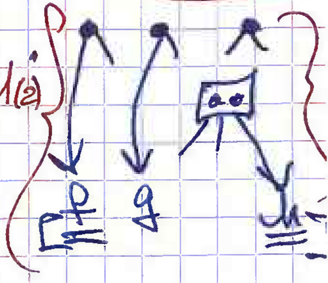
"№2"



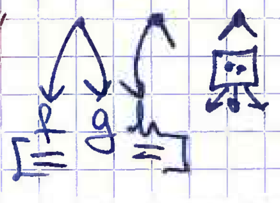
"№1(1)"



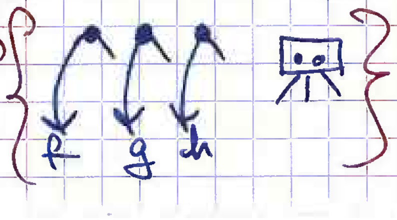
"№1(2)"



"№0(1)"



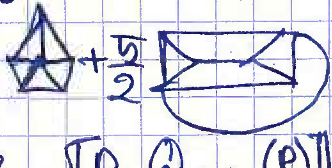
"№0(2)" can-can



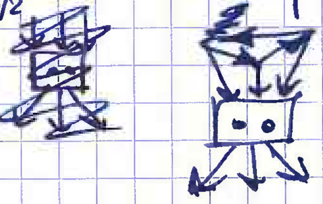
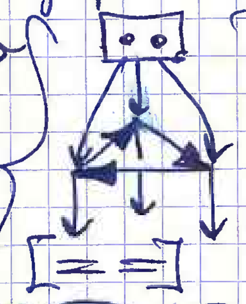
By computer: generate all graphs in ◇(P, Jac(P)) ~~and then~~
w/ undetermined coeffs
equate w/ $[\![P, Q(P)]\!]$ } ⇒ if solution exists, then Q universal

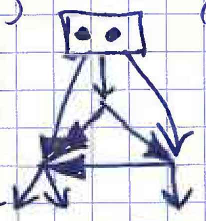
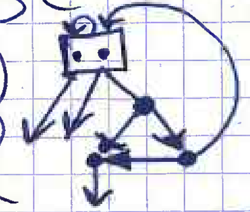
Rem: • ∃ ≥ 1 solution ⟵ Why? [open problem]
• ['96] universal explanation why always ≥ 1 solution
corresp. to topol. identities in Leibniz graphs.
which Leibniz graphs are neighbours ∠ general topol. problem

• γ_5 ⟶ Orient [1412.05259]

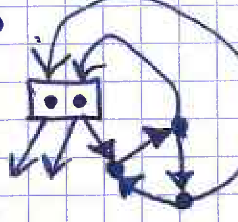
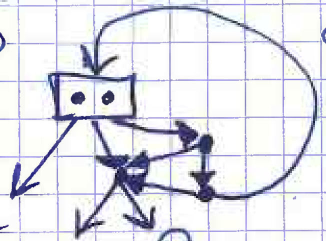


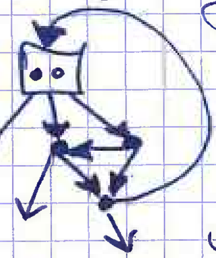
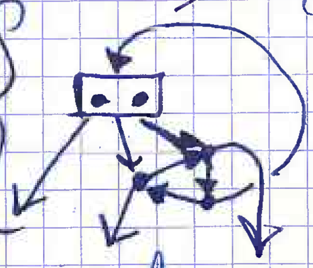
for $[\![P, Q_{1:6/2}(P)]\!]$ ⟵ 39 graphs = 8 graph lin. comb. of ◇

$\Diamond_1(\gamma_3) = 1\{$  $\} + 3\sum_{\sigma\in S_2}(-1)^\sigma\{$  $\}$

$+ 3\sum_{\circlearrowright}\{$  $\} + 3\sum_{\circlearrowright}\{$  $\}$

cyclic permutations

$+ 3\sum_{\circlearrowright}\{$  $\} + 3\sum_{\circlearrowright}\{$  $\}$

$+ 3\sum_{\sigma\in S_3}(-1)^\sigma\{$  $\} + 3\sum_{\sigma\in S_3}(-1)^\sigma\{$  $\}$

Expanding 8 tri-vector graphs ⇒ 210 Kontsevich graphs [1608.01710]
$= 39$ of $[\![P, Q_{1:6/2}(P)]\!]$

another solution
$\Diamond_2 = 112$ tri-vector Leibniz

For γ_5 graph [2:47]

$\Diamond_2 \Leftarrow 8691$ skew-forbug, $\Diamond_1 \Leftarrow 843$

extending infinitesimal deforms to finite deform: graph ↦ each vertex blown by $Q(P)$

$$\varepsilon^k R_k(P) = \text{"}\exp Q(P)\text{"}_k + \underbrace{\Diamond(P, \mathrm{Jac}(P)) + [\![P, \mathcal{Z}(P)]\!]}_{=0 \quad \forall P: [\![P,P]\!]=0}$$

ε^2 term

k-th power series term

we want to find each ε^k term as a graph

"explicitly what garbage we have?" open problem

typically not present "exact term"

§4 [very hard] Open problem: $\forall N^r_{\mathrm{aff}}, \forall P$ ← language of quantum spacetime? [1210.0726]

§7 of Kontsevich quantisation of Poisson manifolds "Formal geo for ★"

$$\left.\begin{array}{l} \mathrm{Or}^{\vec{r}}(\gamma_3) \\ \mathrm{Or}^{\vec{r}}(\gamma_5) \\ \mathrm{Or}^{\vec{r}}(\gamma_7) \\ \mathrm{Or}^{\vec{r}}([\gamma_3, \gamma_5]) \end{array}\right\}$$

in all known examples, ∂_P-exact: vector field ("reparametrisation")

$$Q(P) = [\![P, \overbrace{\exists \mathcal{X}}]\!]$$

? $\mathcal{X}$ not encoded by Kont. oriented graphs (can be dense)

grt ⊆ [Willwacher] ⇒ ∃(?) countably big set

? all trivial (exact) ← ⇔ moving along trajectories of field

changes of coords are non-linear ← flows ~ nonlinear changes of coord

very strong physical implications

graphs in cocycles ~ n^n

Jacobi is abstraction to associativity $\xleftarrow{\text{Kont.}}$ only abstraction

Programs: (AKR, Willwacher, Francis Brown):
allow Or expansion by Leibniz graphs
allow star product ★ expansions [1702.00681 (Exp. Math)]

★ mod $\bar{o}(\hbar^4)$

[1812.xxxxx Francis Brown]

★ mod $\bar{o}(\hbar^{5,6})$ ← Brent Pym, Panzer Eric

relate to polylogarithms

★ mod $\bar{o}(\hbar^7)$ related to $\zeta(3)$

Language of graphs $\mathrm{Or}^{\vec{r}}(\cdot)$ works both for deform of Poisson structures & star products.

! [MK, (1997/2003), field theories, 1705.01774]

Lift from dim<∞ ~~Poisson~~ (N^r_{aff}, P) to geo, where manifold is fiber in bundle over spacetime base. How Hoyle product is quantised

§L3 — Morphism of graph orientation

how, why, how to use in practice?

goal

$$\overrightarrow{\mathrm{Or}}(\cdot)(P) : \gamma \in \ker(d = [\bullet\!\!-\!\!\bullet, \cdot]) \longmapsto \dot{P} = \overrightarrow{\mathrm{Or}}(\gamma)(P) \in \ker(\partial_P = [\![P, \cdot]\!]$$

if $[\![P,P]\!] = 0 \Leftrightarrow P$ is a Poisson bi-vector.

Goal: construct a $\underset{\text{(one)}}{\text{a}}$ solution $\Diamond$ for problem of factorization of cocycle condition

$$[\![P, \overrightarrow{\mathrm{Or}}(\gamma)(P)]\!] = \Diamond(P, \mathrm{Jac}(P))$$

$$\Diamond = \Diamond(\gamma)$$

Ⓢ1

grading shifted lowered by 1

$$\mathrm{End}\left\{ T_{\text{poly}}^{\downarrow[1]}\left(N_{\mathbb{R}}^{r<\infty}\right) \right\}$$

"polyvectors" = multivectors

$|\mathcal{X}| = 1$ but $\overline{\mathcal{X}} = 0$

$|P| = 2$ but $\overline{P} = 1$ shifted

$|\mathrm{Jac}(P)| = 3$ but $\overline{\mathrm{Jac}(P)} = 2$

we study endomorphisms

[1811.07878, 1904.13298]

everything originates from [Kontsevich '96] ← ~~by reading~~ Ⓔ

Willwacher (2010) → (2015) ← Ⓔ

Jost (2013) ← Ⓔ explanation of what goes on Ⓔ

AVK ← Ⓔ

Ex: $[\![\cdot,\cdot]\!]$ Schouten bracket ; $|[\![\cdot,\cdot]\!]| = -1$ but $\overline{[\![\cdot,\cdot]\!]} = 0$

Consider End of form:

$$\theta : T_{\text{poly}}^{d_1} \otimes \cdots \otimes T_{\text{poly}}^{d_k} \longrightarrow T_{\text{poly}}^{d_1+\cdots+d_k+d_r} N^r$$

homogeneity

k-ary ; $\deg\theta = d_r$

Ⓢ2 Insert

$$\underset{\substack{\uparrow\\ i\text{-th argument}}}{\theta_a \vec{\circ}_i \theta_b}\,(p_1, \dots, p_{\underbrace{\#a}_{\#\text{arguments in } \theta_a}+\#b-1}) = \cancel{\theta_b(p_1, \dots, p_{i-1}, \theta_a(p_i, \dots, p_{\#a+\#b-1}}$$

$$= \theta_b(p_1, \dots, p_{i-1}, \theta_a(p_i, \dots, p_{i+\#a-1}), p_{i+\#a}, \dots, p_{\#a+\#b-1})$$

Take

$$\theta_a \vec{\circ} \theta_b = \sum_{i=1}^{k_b = \#b} \theta_a \vec{\circ}_i \theta_b$$

Define

$$\text{"}[\theta_a, \theta_b]\text{"} := \theta_a \vec{\circ} \theta_b - (-1)^{|\theta_a||\theta_b|} \theta_b \vec{\circ} \theta_a$$

graded skew-symmetric by construction

(§3) Extra assumption on Endr based on proto-bracket "[·,·]"

① ← given as skew or we make it skew w/to permutations of arguments, graded shifted Koszul signs

deg-Koszul signs

$$\underset{=}{\epsilon_{p}}(1 \rightleftarrows 2) = (-1)^{(1 \rightleftarrows 2)} \cdot (-1)^{\overline{p_1} \cdot \overline{p_2}}$$

tuple of arguments, graded $\quad = -1$

Ex: Schouten bracket $[\![p_1, p_2]\!] \overset{?}{=} -(-1)^{\overline{p_1}\,\overline{p_2}} [\![p_2, p_1]\!]$

Endomorphism $[\![\cdot,\cdot]\!]$

We need to make "[·,·]" ~~shifted~~ graded-skew : Alternate!

$$[\cdot,\cdot]_{NR} := \mathrm{Alt}(\text{"}[\cdot,\cdot]\text{"}) \leftarrow$$ bi-linear, graded-skew endomorphism.

Nijenhuis-Richardson

~~graphs we follow same logic~~ We mimic procedure for graphs:

Jacobi $([\cdot,\cdot]_{NR})$:

$$[a,[b,c]_{NR}]_{NR} - (-1)^{|a|\cdot|b|}[b,[a,c]_{NR}]_{NR} = [[a,b]_{NR},c]_{NR}$$

(?) $\exists\{\text{Jacobi}([\![\cdot,\cdot]\!]) \overset{?}{=} 0\} \overset{?}{\Longrightarrow} [\pi_S,\pi_S]_{NR} = 0$

$$\pi_S(P,Q) := (-1)^{\overline{P}} [\![P,Q]\!]$$

Def of Schouten: local function F, G

reduced ?

$$\Delta_{BV}(F \cdot G) = \vec{\Delta}(F)\cdot G + \underbrace{(-1)^{F}[\![F,G]\!]}_{\pi_S(F,G)} + (-1)^{F} F \cdot \vec{\Delta} G$$

Schouten { measures deviation of BV Laplacian being a derivation (Leibniz rule) }

[1312.1262, 1210.0726]

Schouten w/ P is differential ⇝ cohomology

NR w/ $a = b = \pi_S$: $\underbrace{2[a,[a,\cdot]_{NR}]_{NR}}_{=\text{differential squared}} = [\underbrace{[a,a]_{NR}}_{=[\pi_S,\pi_S]_{NR}=0},\cdot]_{NR} = 0$

$$\partial^2_{\pi_S} = ([\pi_S,\cdot]_{NR})^2 = 0 \rightsquigarrow \text{cohomology}$$

$$\partial_{\pi_S}(\cdot) = [\pi_S,\cdot]_{NR}$$

Corr: $\mathrm{End}^{\partial x,\partial x}_{\text{skew}}\left\{T^{\downarrow[1]}_{\text{poly}}\left(N^{r<\infty}_{\text{aff}}\right)\right\}$ has a differential ∂_{π_g}

homogeneities

§4a Graphs & endomorphisms:

Graphs	Endomorphisms
Graph$(\gamma, \underset{\text{wedge ordering}}{E(\gamma)})$	~~Skew~~ Endo Θ
Insert $\gamma_a \vec{\circ}_i \gamma_b$; $\gamma_a \vec{\circ} \gamma_b$	Insert $\Theta_a \vec{\circ}_i \Theta_b$; $\Theta_a \vec{\circ} \Theta_b$
Bracket $[a,b] = a\vec{\circ} b - (-1)^{\#E(a)\#E(b)}\, b\vec{\circ} a$ ↳ Lie alg. structure $([a,b], E(a)\wedge E(b))$	proto Bracket "$[\Theta_a,\Theta_b]$" $\xrightarrow{\text{Alt}}$ $[\Theta_a,\Theta_b]_{NR}$
Jacobi $([\cdot,\cdot])$	Jacobi $([\cdot,\cdot]_{NR})$
Edge $\bullet\!\!-\!\!\bullet$ $([\bullet\!\!-\!\!\bullet, \bullet\!\!-\!\!\bullet] = 0 \in \mathrm{Gra})$	Schouten $\pi_g = \pm [\![\cdot,\cdot]\!]$ $[\pi_g,\pi_g]_{NR} = 0 \in \mathrm{End}$
Differential $d = [\bullet\!\!-\!\!\bullet, \cdot]$	Differential $\partial_{\pi_g} = [\pi_g, \cdot]_{NR}$

We now construct a map Gra → End that respects structures (Lie algebra morphis ⟺ respects d, ∂).

$$\begin{tikzcd}
\gamma \arrow[r, mapsto, "\Theta_{\vec{r}}"] \arrow[d, mapsto, "d"'] & \Theta_r(\gamma) \arrow[d, mapsto, "\partial_{\pi_g}"] \\
{[\bullet\!\!-\!\!\bullet, \gamma]} \arrow[r, mapsto, "\Theta_{\vec{r}}"'] & {[\pi_g, \Theta_r(\gamma)]}
\end{tikzcd}$$

diagram commutes. (CD)

§4b

$$\underset{\text{End}}{\boxed{\overrightarrow{\Theta r}(\gamma)}}\,\overbrace{(p_1,\ldots,\underset{\substack{\| \\ \#\text{arg} = \cancel{\#\text{Vert}} = n}}{p_k})}^{(x,\xi)}(x,\xi) :=$$

produces non-skew endomorphisms

$$= \mathrm{mult}\left\{\left(\vec{\prod_{e_{ij}\in E(\gamma)}} \vec{\Delta}_{ij}\right)(p_1\otimes\cdots\otimes p_k)\right\}(x,\xi)$$

edge operator

$$\vec{\Delta}_{ij} : \overset{i}{\bullet}\!\!\longrightarrow\!\!\overset{j}{\bullet} \longmapsto \left(\overset{i}{\bullet}\xrightarrow[\ell]{}\overset{j}{\bullet} + \overset{i}{\bullet}\xleftarrow[\ell]{}\overset{j}{\bullet}\right)$$

$$\vec{\Delta}_{ij} = \sum_{\ell=1}^{r=\dim N^r}\left(\frac{\vec{\partial}}{\partial x^{\ell}_{(j)}} \circ \frac{\vec{\partial}}{\partial \xi^{(i)}_{\ell}} + \frac{\vec{\partial}}{\partial \xi^{(j)}_{\ell}} \circ \frac{\vec{\partial}}{\partial x^{\ell}_{(i)}}\right)$$

diff w/to tail $\underset{=i}{}$ ξ, head $\underset{=j}{}$ x

diff w/to tail ξ head $\underset{=i}{}$ x

Claim: $\gamma_a \vec{\circ} \gamma_b \xmapsto{\overrightarrow{or}} \overrightarrow{or}(\gamma_a) \vec{\circ} \overrightarrow{or}(\gamma_b)$ ("Leibniz rule")

endomorphisms encoded by graphs

Induce (skew)

$$\mathcal{O}\vec{r} : (\gamma, E(\gamma))^{\mathrm{Gra}} \longmapsto \mathrm{End}^{\mathrm{d}x,\,\mathrm{d}k}_{\mathrm{skew}} \left\{ T^{\downarrow[1]}_{\mathrm{poly}} \left(N^{r<\infty}_{\underline{\underline{\mathrm{aff}}}} \right) \right\}$$

essential that N^r is affine

to get Jacobians

$i\,\bullet\!\xrightarrow[J^{-1}\qquad J]{}\! j$

coordinate-free

[cannot depend on X]

[1705.01777 diagonally]

Ex: $\mathcal{O}\vec{r}(\bullet\!\!-\!\!\bullet) \overset{?}{=} \pi_{\mathcal{S}}$ {not written anywhere?}

the argument influences the endomorphism $x \mapsto \overset{\mathrm{Bach}}{[x \otimes \xi]}(x)$; for us:

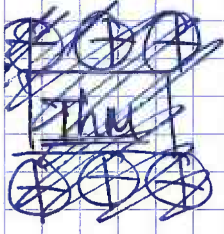
$\overset{i}{\bullet}\!\longrightarrow j$ + $\longleftarrow\!\bullet$ } we allow to consider & able to encode this class of endomorphisms

endr arg endr arg

Rem: if all multivectors encoded by x, ξ ⇒ Alt redundant:

ordered arguments $p_1 \otimes \cdots \otimes p_k \rightsquigarrow \xi$ killed by edge operator

how: specified by the graph.



MAIN (climax)

Thm: $\mathcal{O}\vec{r}$ is a Lie-algebra-morphism:

$$\mathcal{O}\vec{r}([\gamma, \beta]) = [\mathcal{O}\vec{r}(\gamma), \mathcal{O}\vec{r}(\beta)]_{NR}$$

proof not written

"nothing to prove"

Corr: $\mathcal{O}\vec{r}([\bullet\!\!-\!\!\bullet, \gamma]) = \mathcal{O}\vec{r}(\mathrm{d}(\gamma))$ ~~$= [\pi_{\mathcal{S}}, \mathrm{d}(\gamma)]_{NR} = \partial_{\pi_{\mathcal{S}}}(\mathrm{d}\gamma)$~~

$$= [\pi_{\mathcal{S}}, \mathcal{O}\vec{r}(\gamma)]_{NR} = \partial_{\pi_{\mathcal{S}}}(\mathcal{O}\vec{r}(\gamma))$$

⇒ Diagram (CD) commutes

"the edge strikes back"

Parasitic algebraic mismatches from $1/k!$ ← insignificant.

Gives a solution for factorization problem ← just one (?!) solution

18

(35) If all graph vertices restricted to Poisson bi-vectors ⇝ Corr.?

<u>Universal symmetries</u>: $\dot{P} = \overrightarrow{\mathrm{Or}}(\gamma \in \ker d)(P)$ and $\Diamond$.

Supp. $\#\mathrm{Vert}(\gamma) = n$, $\#\mathrm{Edges}(\gamma) = 2n-2$.

$|\deg(\overrightarrow{\mathrm{Or}}(\gamma))| = -\#E(\gamma)$

every edge differentiates out one ξ ← placeholder for argument of multivector.

$$\overrightarrow{\mathrm{Or}}(\underbrace{[\bullet\!\!-\!\!\bullet, \gamma]}_{\gamma \text{ cocycle} \Rightarrow\ = 0})(\underbrace{P, \ldots, P}_{n+1}) =$$

$$= \Big(\pi_S \vec{\circ}\, \overrightarrow{\mathrm{Or}}(\gamma)\Big)(P, \ldots, P) - \underbrace{(-1)^{\overset{=-1}{|\pi_S|} \cdot \overset{\text{even}}{(2n-2)}}}_{=\oplus}\Big(\ldots$$

$$\ldots \Big(\overrightarrow{\mathrm{Or}}(\gamma)\, \vec{\circ}\, \pi_S\Big)(P, \ldots P) =$$

$$= \boxed{\overrightarrow{\mathrm{Or}}(\gamma)}\big(\pi_S(P,P), P, \ldots, P\big) + \cdots + \boxed{\overrightarrow{\mathrm{Or}}(\gamma)}\big(P, \ldots, P, \pi_S(P,P)\big) -$$

$$- \pi_S\Big(\overrightarrow{\mathrm{Or}}(\gamma)(\underbrace{P, \ldots, P}_{n = \#\mathrm{Vert}(\gamma)}), \overset{\uparrow}{P}\Big) - \pi_S\Big(P, \overrightarrow{\mathrm{Or}}(\gamma)(P, \ldots, P)\Big).$$

bi-vectors

Lemma (?): $\overrightarrow{\mathrm{Or}}(\underset{\substack{\uparrow\\ \text{zero graph}}}{0}) = \underset{\text{zero orgraph}}{\ominus}$

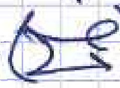
$$\textcircled{\Rightarrow}\ 0 = \overrightarrow{\mathrm{Or}}(\gamma)\big(-[\![P,P]\!], P, \ldots, P\big) + \cdots + \overrightarrow{\mathrm{Or}}(\gamma)\big(P, \ldots, P, -[\![P,P]\!]\big)$$

$$+ [\![\underbrace{\overrightarrow{\mathrm{Or}}(\gamma)(P, \ldots, P)}_{\text{bi-vector}}, P]\!] + [\![P, \underbrace{\overrightarrow{\mathrm{Or}}(\gamma)(P, \ldots, P)}_{\text{bi-vector}}]\!]$$

graded ~~skew~~ $-(-1)^{(2-1)(2-1)} = +$

$$\Rightarrow [\![P, \underbrace{\overrightarrow{\mathrm{Or}}(\gamma)(P, \ldots, P)}_{=: Q(P) \text{ bi-vector}}]\!] = \frac{1}{2}\Big\{\overrightarrow{\mathrm{Or}}(\gamma)([\![P,P]\!], P, \ldots, P) + \cdots + \overrightarrow{\mathrm{Or}}(\gamma)(P, \ldots, P, \underbrace{[\![P,P]\!]}_{\mathrm{Jac}(P)})\Big\}$$

$$= \partial_P(Q(P)) \qquad\qquad = \Diamond(P, \mathrm{Jac}(P))$$

=  {Leibniz orgraphs}

$$\partial_P(Q(P)) = \Diamond(P, \mathrm{Jac}(P))$$

NB! ↑ one solution $\Diamond_1$. (∃ more)

Rem (?): If $\gamma = d(\beta) \Rightarrow \overrightarrow{\mathrm{Or}}(d(\beta))(P) = 2[\![P, \overrightarrow{\mathrm{Or}}(\beta)(P)]\!]$

$\#E = 2n-2$, $\#V = n$ $\qquad$ $\#E = 2n-3$, $\#V = n-1$

1-vector $\mathcal{X}(P)$ encoded by Kontsevich

+ ⟨improper terms with $[\![P,P]\!]$⟩ "garbage" = 0 in Poisson

⇒ Poisson-cohomology-exact $= \partial_P(\mathcal{X}(P)) + $ 

(36)

Ex:	γ_3	γ_5	γ_7	$[\gamma_3, \gamma_5]$
#Vert(γ):	4	6	8	9=4+6-1
#E(γ):	6	10	14	16
#graphs in cocycle	1	2 (wheel+prism)	46	— (cannot remember) [Willwacher]
#Kont. orgraphs in $Q(P)$	$3_{(2)}$ ↑ due to a skew-symmetrization	$167_{(91)}$	$34185_{(-)}$	$-_{(-)}$
#Kont. orgraphs in cocycle cond. $[\![P, Q(P)]\!]$	$39_{(27)}$	$3495_{(-)}$	$1\,003\,611_{(-)}$	$-_{(-)}$
#skew Leibniz orgraphs in $\Diamond(P, \mathrm{Jac}(P))$	$8_{(27)}$ ↑ after cyclication $\#\Diamond_2 = 112$	843 $\#\Diamond_2 = 8691$	293 694	—

(37) Tetrahedral flow — [1904.13293] ←

"rule of signs" 1st $\downarrow$ $\boxed{+1}$ $-3 \{\text{flow}\}\{\text{sign}\}$

(?) sign between $1 : 6/2$?
specified by $\vec{\Theta}_E$

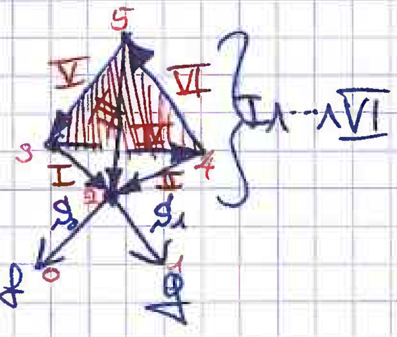
$(-1)^{\sigma_E}$

$\gamma_3 =$ 

nontrivial cocycle

$$\Big(\overset{2}{\overset{01}{\mathcal{S}_0 \mathcal{S}_1}} \quad \overset{3}{\overset{24}{\mathrm{I}\ \mathrm{IV}}} \quad \overset{4}{\overset{25}{\mathrm{II}\ \mathrm{VI}}} \quad \overset{5}{\overset{23}{\mathrm{III}\ \mathrm{V}}} \Big) = \sigma_E (\mathcal{S}_0 \wedge \mathcal{S}_1 \wedge \mathrm{I} \wedge \cdots \wedge \mathrm{VI})$$

ordering of indices



every vertex — source of 2 arrows

4 — # choices of vertex for $\overset{\wedge}{f\ g}$

2 — face ▲ clockwise/counter

$$\underbrace{\begin{matrix} 4 \\ \cdot \\ 2 \\ \parallel \\ 8 \end{matrix}}$$

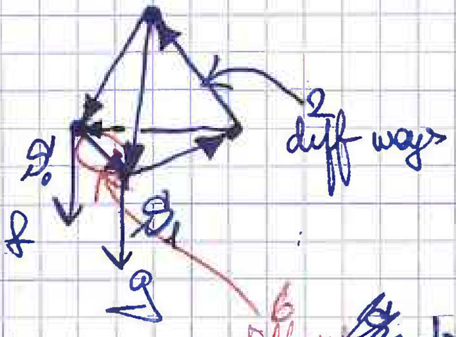
have signs!

Claim: all 8 contribute $\oplus$ from (?)

$\sigma_E \in S_{6=\#E}$ under orgraph $\cong$

$\gamma_3 \neq 0$

For $\gamma_3 = 0$ the value is = exactly half.



6 different ways to choose; 2 ways to orient

$2 \cdot \underline{6} \cdot \underline{2} = 24$ all $\oplus$

$\Rightarrow$ proportion $8 : 24 = 1 : 3 = 1 : 6/2$

Rules for sign:

1. $\overset{\frown}{S_0 A}\ \ \overset{\frown}{S_1 B}\ ,\ A < B \Rightarrow -$
2. $\Lambda \rightleftarrows \Lambda \quad \mathrm{sign}(\Gamma_2) = (-1)^{\#\text{crossings in}}\ \mathrm{sin}(\Gamma_1)$

   $\Lambda \rightleftarrows \Pi \quad -\ \ ''$

   $\Pi \rightleftarrows \Pi \quad ''$

Implications: we are sitting on affine manifolds [physically]

flows well def. on affine; not necc. symplectic, even dim

∞ many flows by Teichmüller

Relevant flows: open explorations.

Why: cohomology Poisson trivial $\Rightarrow$ model smooth manifold structures