Finite-dimensional polynomial algebras with Wronskians over $\mathbb{R}^d$ as N-ary Lie brackets: beyond $\mathfrak{sl}(2)$ and the Witt algebra

Markuss G. Ķēniņš
University of Groningen, the Netherlands

Joint work with Arthemy V. Kiselev

arXiv: 2605.27305

Monday, 13th of July, 2026

Motivation: higher structures in physics

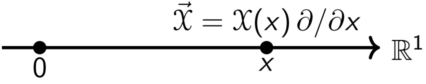
- **Lie groups $\leftrightarrows$ algebras:** QFT, gauge field theory
  Example: $\mathfrak{sl}(2)$ is the Lie algebra of the Lorentz group
- **Higher Lie structures:** string theory, gravitational physics, particles of higher spin, BRST in Batalin–Vilkovisky, closed string field theory, non-commutative Chern–Simmons and Yang–Mills theories, star products

  [Stasheff: "L_∞ and A_∞ structures", 2019]
- **Higher arity differential structures $\leftrightarrows$ Wronskian determinants:** realisation of $\mathfrak{sl}(2)$, calculation of topological charge of Skyrmions, alternation of differential operators

  [Dzumadil'daev 2002, Kiselev 2004 & 2025, us 2026]
- **<u>Wronskians realise higher arity Lie structures</u>**

Introduction: Lie algebras $\leftarrow$ Wronskian determinants

Example: $\mathfrak{sl}(2)$ realisation by Wronskians



$$[\vec{\mathcal{X}}, \vec{\mathcal{Y}}] = \vec{\mathcal{X}} \circ \vec{\mathcal{Y}} - \vec{\mathcal{Y}} \circ \vec{\mathcal{X}} = \begin{vmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{X}' & \mathcal{Y}' \end{vmatrix} \frac{\partial}{\partial x} = W^{0,1}(\mathcal{X}, \mathcal{Y})\,\partial/\partial x$$

$$\begin{aligned} \mathfrak{sl}(2) &\simeq \big(\langle \vec{\mathbf{e}} = 1\,\partial/\partial x, \quad \vec{\mathbf{f}} = -x^2\,\partial/\partial x, \quad \vec{\mathbf{h}} = -2x\,\partial/\partial x\rangle, [\cdot,\cdot]\big) \\ &\simeq \big(\langle 1, -x^2, -2x\rangle, W^{0,1}\big) \end{aligned}$$

Higher order Wronskians

Put $N = 2p$; alternate the pth order differential operators

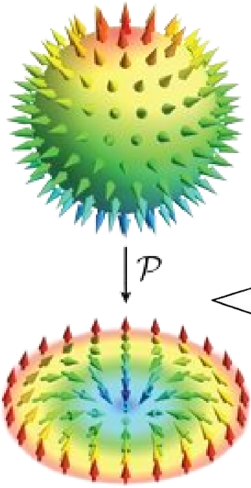
$$\mathcal{W}_j = w_j(x)\,\partial^p/\partial x^p, \qquad j = 1, \ldots, N$$

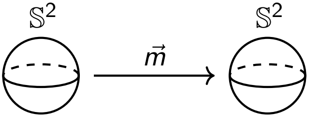
$$\sum_{\sigma\in S_N} (-)^\sigma (\mathcal{W}_{\sigma(1)} \circ \cdots \circ \mathcal{W}_{\sigma(N)}) = \text{const} \cdot \underbrace{\begin{vmatrix} w_1 & \cdots & w_N \\ w_1' & \cdots & w_N' \\ \vdots & & \vdots \\ w_1^{(N-1)} & \ldots & w_N^{(N-1)} \end{vmatrix}}_{= W^{0,1,\ldots,N-1}(w_1,\ldots,w_N)} \frac{\partial^p}{\partial x^p}$$

Wronskian of differential order $N-1$ over $\mathbb{R}^1$: $W^{0,1,\ldots,N-1}$

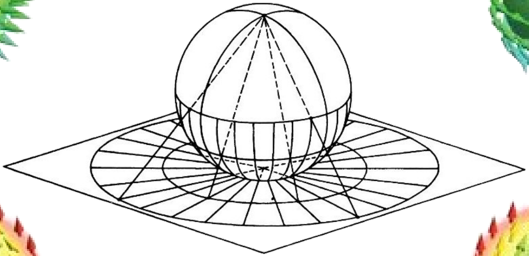
[Kiselev, **math/0410185**, Kian Shah & Kiselev **2605.11137**]

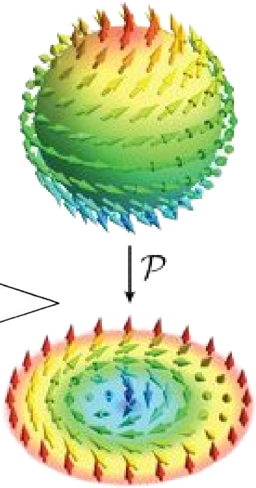
Skyrmions $\leftrightarrows$ homology of spheres $\leftarrow$ Wronskians over $\mathbb{R}^d$



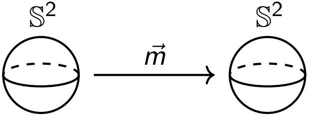


$$\vec{m} : \mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2 \to \mathbb{S}^2 \subsetneq \mathbb{R}^3$$





Skyrmions $\leftrightarrows$ homology of spheres $\leftarrow$ Wronskians over $\mathbb{R}^d$



$$\vec{m} : \mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2 \to \mathbb{S}^2 \subsetneq \mathbb{R}^3$$

$$\vec{m} = (m_1, m_2, m_3)$$

sphere homology $\quad H_2(\mathbb{S}^2) \cong \mathbb{Z}$

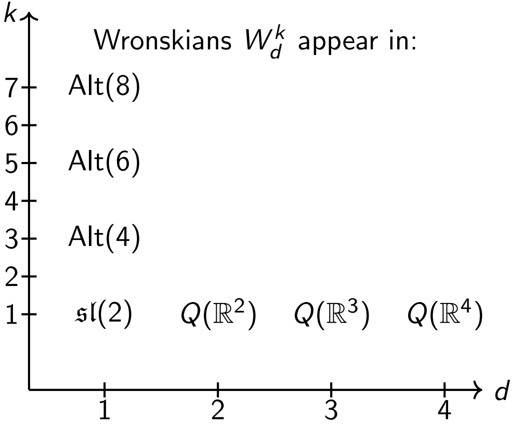
$\vec{m} \leftrightarrows Q_{\vec{m}}$ topological charge

$$Q = \int_{\mathbb{S}^2} \vec{m}^* \omega = \frac{1}{4\pi} \int_{\mathbb{R}^2} \underbrace{\det[\vec{m}, \partial_x \vec{m}, \partial_y \vec{m}]}_{= W^{k=1}_{d=2}(m_1, m_2, m_3)} \, dxdy$$

Complete generalised Wronskians

Wronskian over $\mathbb{R}^d$ of differential order $k \geqslant 1$: W_d^k

$d = 2 = k$: $\quad W_{d=2}^{k=2}(f, g, ...)$ has rows $\quad f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$



Higher Lie structures

Definition: N-ary Lie algebra

$$\mathcal{A} = (V, [\cdot, ..., \cdot] : V^N \to V)$$

multilinear, totally antisymmetric, Schlessinger–Stasheff Jacobi

$$\sum_{\sigma \in S_{2N-1}} (-)^\sigma [[a_{\sigma(1)}, ..., a_{\sigma(N)}], a_{\sigma(N+1)}, ..., a_{\sigma(2N-1)}] = 0$$

Generalised Wronskians satisfy the Schlessinger–Stasheff identities.

[Dzumadil'daev 2002, Kiselev 2004]

Example:

$$\mathcal{A} = (\Bbbk[x^1, ..., x^d],\ W_d^k)$$

space of all polynomials with the Wronskian as bracket
is an *infinite-dimensional* N-ary Lie algebra

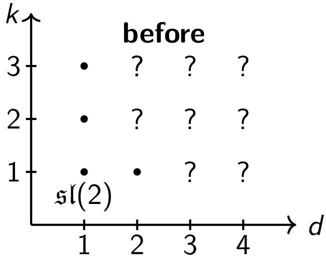
Previously known **finite-dimensional** N-ary Lie algebras

- $\mathfrak{sl}(2) \simeq (\langle 1, x, x^2\rangle,\ [\cdot,\cdot] = W^{0,1})$
- $\mathcal{A} = (\langle 1, x^1, x^2, ..., x^N\rangle,\ W^{0,1,...,N-1})$

$$W^{0,1...,N-1}\Big(1, \frac{x^1}{1!}, ..., \frac{\widehat{x^\ell}}{\ell!}, ..., \frac{x^N}{N!}\Big) = \frac{x^{N-\ell}}{(N-\ell)!}$$

- $\mathcal{A} = (\langle 1, x, y, xy\rangle,\ [\cdot,\cdot,\cdot] = W^{k=1}_{d=2})$

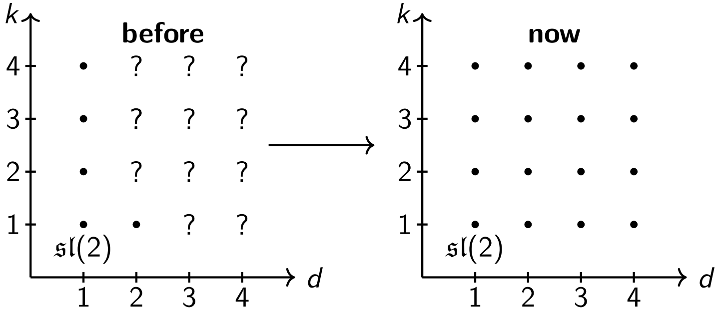
$$[1,x,y] = 1 \quad [1,x,xy] = x \quad [1,y,xy] = -y \quad [x,y,xy] = -xy$$



[Kiselev, **math/0410185**]

Research problem

Fix any base dimension $d = \dim \mathbb{R}^d$ and any differential order $k \geqslant 1$ of the Wronskian W_d^k; *do there exist* **finite-dimensional** subspaces of polynomials over $\mathbb{R}^d$ closed under W_d^k?



[2605.27305]

Main theorems: all* finite-dimensional N-ary Lie algebras with Wronskians as brackets

All polynomials up to degree k: $\mathbb{k}_{k=2}[x,y] = \langle 1, x, y, x^2, xy, y^2 \rangle$

Theorem (Main-1): For any $k, d \geqslant 1$ and any homogeneous polynomial $p(\boldsymbol{x})$ of degree $k+1$,

$$\mathcal{A} = (\mathbb{k}_k[x^1, ..., x^d] \oplus \langle p(\boldsymbol{x}) \rangle,\ W_d^k)$$

is an N-ary Lie algebra with the generalised Wronskian W_d^k as the N-ary bracket, $N = \binom{d+k}{k}$. Here $\dim \mathcal{A} = N + 1 < \infty$.

Theorem (Main-2): If $\mathcal{A} \supsetneq \mathbb{k}_k[x^1, ..., x^d]$ is **not** of the form above, then $\dim \mathcal{A} = \infty$.

[Ķēniņš and Kiselev, **2605.27305**]

Structure constants

Put $\boldsymbol{x}^{\vec{\boldsymbol{r}}} = (x^1)^{r_1} \cdots (x^d)^{r_d}$. For any monomial $a(\boldsymbol{x}) \in \Bbbk[x^1, \ldots, x^d]$:

$$W_d^k\Big(1, \ldots, \underbrace{a(\boldsymbol{x})}_{\boldsymbol{x}^{\vec{\boldsymbol{r}}}}, \ldots, \frac{(x^d)^k}{k!}\Big) = \frac{(-)^{k-r}}{(k-r)!} \prod_{\ell=r+1}^{k} (\deg a - \ell) \cdot \frac{\partial^{|\vec{\boldsymbol{r}}|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}}}\, a(\boldsymbol{x})$$

Example: $\mathcal{A} = (\langle 1, x, y, z, p(\boldsymbol{x}) = xy + yz + zx \rangle,\ W_{d=3}^{k=1})$

$$[1, x, y, z] = 1, \qquad [x, y, z, p] = p$$

$$[1, x, y, p] = x + y, \quad [1, x, z, p] = -x - z, \quad [1, y, z, p] = y + z$$

Perfect ? Algebras $\mathcal{A}$

Definition: $\mathcal{A} = \big(V, [\cdot, ..., \cdot]_N\big)$ *perfect* if $\operatorname{span}_{\Bbbk}[\mathcal{A}, ..., \mathcal{A}]_N = \mathcal{A}$

Lie: semi-simple $\Longrightarrow$ perfect.

Examples:

- $\mathfrak{sl}(2) \simeq \big(\langle 1, x, x^2\rangle, W^{0,1}\big)$
- $\mathcal{A} = \big(\langle 1, x, x^2, ..., x^N\rangle, W^{0,1,...,N-1}\big)$
- $\mathcal{A} = \big(\langle 1, x, y, xy\rangle, W^{k=1}_{d=2} = \mathbf{1} \wedge \partial/\partial x \wedge \partial/\partial y\big)$
- $\mathcal{A} = \big(\langle 1, x, y, z, xy + yz + zx\rangle, W^{k=1}_{d=3}\big)$

Non-examples:

- $\mathcal{A} = \big(\langle 1, x, y, x^2\rangle, W^{k=1}_{d=2}\big)$
- $\mathcal{A} = \big(\langle 1, x, y, y^2\rangle, W^{k=1}_{d=2}\big)$
- $\mathcal{A} = \big(\langle 1, x, y, z, \textbf{any degree 2 monomial}\rangle, W^{k=1}_{d=3}\big)$

Perfect algebras require polynomials

Proposition: If $a(\boldsymbol{x})$ **monomial** of degree $k+1$:

$\mathcal{A} = \left(\mathbb{k}_k[x^1, ..., x^d] \oplus \langle a(\boldsymbol{x})\rangle, W_d^k\right)$, is perfect only if

$$(i) \quad d = 1 \qquad \text{or} \qquad (ii) \quad d = 2 \quad \text{and} \quad k = 1$$

Problem: Does there exist a *polynomial* $a(\boldsymbol{x})$ s.t. $\mathcal{A}$ is perfect?

Over $k = 1$ and any d not divisible by 4, yes:

$$a(\boldsymbol{x}) = x^1x^2 + x^2x^3 + \cdots + x^{d-1}x^d + x^dx^1$$

for instance, $a(\boldsymbol{x}) = xy + yz + zx$

Witt algebra and its generalisations

Witt algebra: (over $\mathbb{C}$) Virasoro algebra with zero central charge

$$L_n = a_n \frac{\partial}{\partial z}, \quad a_n = z^{n+1}$$

$$[L_n, L_m] = (m-n)L_{n+m} \ \leftrightarrows \ [a_n, a_m] = W^{0,1}(a_n, a_m) = (m-n)a_{n+m}$$

Deformations over higher arity: $[\cdot, ..., \cdot] = W^{0,1,...,N-1}$, $a_n = z^{n+N/2}$

$$[a_{n_1}, ..., a_{n_N}] = \prod_{1 \leqslant i < j \leqslant N} (n_j - n_i) \cdot a_{n_1 + \cdots + n_N}$$

Generalisations over $\mathbb{C}^d$: *Example:* take $d = 2$ and $k = 1$; put $\vec{\boldsymbol{n}} = (n_1, n_2)$, similarly $\vec{\boldsymbol{m}}$ and $\vec{\boldsymbol{\ell}}$. Then $a_{\vec{\boldsymbol{n}}} = z^{n_1+3/2}\, w^{n_2+3/2}$ satisfy

$$[a_{\vec{\boldsymbol{n}}}, a_{\vec{\boldsymbol{m}}}, a_{\vec{\boldsymbol{\ell}}}] = \begin{vmatrix} 1 & 1 & 1 \\ n_1 & m_1 & \ell_1 \\ n_2 & m_2 & \ell_2 \end{vmatrix} \cdot a_{\vec{\boldsymbol{n}}+\vec{\boldsymbol{m}}+\vec{\boldsymbol{\ell}}}$$

Factorisations of Vandermonde determinants

Ordinary Vandermonde:

$$\det \begin{vmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_N \\ \vdots & & \vdots \\ m_1^{N-1} & \cdots & m_N^{N-1} \end{vmatrix} = \prod_{1 \leqslant i < j \leqslant N} (m_j - m_i)$$

Generalisation over $\mathbb{R}^d$: $\partial^3/\partial x^2 \partial y \quad \longrightarrow \quad (n_1, n_2) \mapsto n_1^2\, n_2$

$$\det \mathrm{Van}_{d=2}^{k=1}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \det \begin{pmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

Does not factorise in general. [Francis Brown, *J. Alg.*, 2025]

We factorise them when $N-1$ or $N-2$ arguments are fixed.

Factorise general Vandermonde: differential operators

From structure theorem the R.H.S. is:

$$\frac{(-)^{k-r}}{(k-r)!}\prod_{\ell=r+1}^{k}(\deg a-\ell)\cdot\frac{\partial^{|\vec{\boldsymbol{r}}|}}{\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}}}a(\boldsymbol{x}),$$

Put $p(\boldsymbol{x})=(x^1)^{n_1}\cdots(x^d)^{n_d}$ and $q(\boldsymbol{x})=(x^1)^{m_1}\cdots(x^d)^{m_d}$.

$$\{p,q\}_{\widehat{x}}:=\{(\deg p-1)m_1-(\deg q-1)n_1\}\frac{pq}{x}$$

$$\{p,q\}_{\widehat{(x)^2}}:=\{(\deg p-1)(\deg p-2)\,m_1(m_1-1)-$$
$$-(\deg q-1)(\deg q-2)\,n_1(n_1-1)\}\frac{pq}{(x)^2},$$

Problem: Write the formulas as antisymmetric differential operators acting on monomials p and q!
Extend to more monomials: $p,q,s,\ldots.$ *Big source of skew operators.*

Conclusions and outlook

Conclusions:

1. Classified all finite-dimensional polynomial algebras with Wronskians as N-ary Lie brackets: *analogues of* $\mathfrak{sl}(2)$ *of higher arity over higher base* $\mathbb{C}^d$*.*
2. Generalised the Witt algebra from field theory to higher (complex) base dimension $\mathbb{C}^d$ and differential order.
3. Factorised the generalised Vandermonde determinants over $\mathbb{C}^d$ in certain base cases.

Open problems: [**5** problems in **2605.27305**]

1. Can the assumption that $\mathcal{A} \supsetneq \Bbbk_k[x^1, ..., x^d]$ be dropped in the 2nd main Theorem?
2. Do there exist topological invariants, which use differential operators of higher order: Wronskians with $k > 1$? If so, what are the applications to particle or solid-state physics?
3. What are the physical uses of the generalised Witt algebras?