university of groningen / faculty of science and engineering / mathematics and applied mathematics

WRONSKIAN DETERMINANTS OVER $\mathbb{R}^d$ AS N-ARY LIE BRACKETS: IDENTITIES, ASYMPTOTICS, AND TOPOLOGICAL CONNECTIONS TO MAGNETIC SKYRMIONS

MARKUSS GUSTAVS ĶĒNIŅŠ

Thesis submitted for the award of the degrees

Bachelor of Science in Mathematics

and

Bachelor of Science in Physics

to the

Faculty of Science and Engineering

of the

University of Groningen

in

Groningen, The Netherlands

on the

10th of July, 2026

Mathematics supervisor: Dr hab. ARTHEMY V. KISELEV

Physics supervisor: Prof. Dr MAXIM V. MOSTOVOY

Updated on the 1st of September, 2026

ABSTRACT. Lie structures are fundamental algebraic tools in physics: the brackets measure the (non-)commutativity of two symmetries or flows. They are deformed to higher (homotopy) Lie structures in certain quantum field theories (e.g. topological Yang–Mills), where higher-arity brackets encode collective interactions. Wronskian determinants realise higher Lie structures: $\mathfrak{sl}(2)$ by vector fields with polynomial coefficients, and its deformations over the real line. Topological constructions from condensed matter physics and QCD – Skyrmion topological charge – are realised in Cartesian coordinates by Wronskians over $\mathbb{R}^d$, which also deform Lie structures to higher base dimension. Following Kirillov, Kontsevich, and Molev, the problem is to describe the growth under iteration of brackets of N-ary Lie algebras built from polynomials, with Wronskians over $\mathbb{R}^d$ as brackets: in particular, what are all the *finite-dimensional* algebras (that do not grow at all)?

We answer this question by proving new determinant identities; to spot them and conjecture that they always hold, we use computer algebra. Subject to mild assumptions about their structure, we find *all* finite-dimensional polynomial N-ary Lie algebras with Wronskians over $\mathbb{R}^d$ as brackets; they pattern upon $\mathfrak{sl}(2)$. We also find strong homotopy deformations and generalisations over $\mathbb{C}^d$ of the Witt algebra from CFT. Finally, when the algebra can grow, we asymptotically bound the growth in degree by the N-bonacci numbers.

We pronounce 10 open problems, which stand immediately beyond our results – base steps towards larger problems in mathematical physics. What we develop here are aimed to serve as mathematical tools for next-generation models in gauge field theory, especially to study collective effects and topological properties.

CONTENTS

1. Overview 4
1.1. Relevance and scientific context 4
1.2. Skyrmions and Wronskians 5
1.3. Local embedding 7
1.4. Research problem 8
1.4.1. Problem about finite dimension 8
1.4.2. Problem about polynomial degree growth 8
1.4.3. Research subject 8
1.4.4. Research object 8
1.4.5. Research strategy 9
1.5. Scientific novelty of results 9
1.6. Open problems 10
1.7. Practical significance of results 13
1.8. Personal contribution 14
1.9. Approbation of results 14
1.9.1. International conference 14
1.9.2. Seminar 14
1.9.3. Talks by the mathematics supervisor 14
1.10. Structure of the thesis 14
1.11. Conclusions 14
1.12. Publications 15
References 15

Research articles 18
2 Explicit class of finite-dimensional polynomial algebras with Wronskians over $\mathbb{R}^d$ as N-ary Lie brackets: beyond $\mathfrak{sl}(2)$ (44 p.) 19
3 Iterate Wronskians over $\mathbb{R}^d$ as N-ary brackets on $\mathbb{R}[x^1, ..., x^d]$: the N-bonacci numbers bound the highest total degrees (13 p.) 63

Back matter 76
Acknowledgements 76
Anotācija 76
Samenvatting 77
Summary for laymen 77

1. Overview

1.1. **Relevance and scientific context.** Lie groups and Lie algebras are fundamental in gauge field theory, in particular in the magnetic or spin systems and in many other models of (liquid) crystals and condensed matter [19].

A purely analytic approach – from the $\mathbb{Z}/2\mathbb{Z}$-graded supermathematics – to Lie algebra structures encodes them in terms of the parity-odd, homological vector fields: $\vec{\mathcal{Q}}^2 = 0$. This is a very commonly used language: it is practised, for example, in the AKSZ-formalism for string theory [1]. By preserving the Jacobi identity $\vec{\mathcal{Q}}^2 = 0$ in a given $\mathbb{Z}/2\mathbb{Z}$-graded setting, one deforms the Lie algebra at hand to a homotopy Lie algebra structure, now with multilinear anti-symmetric brackets of arities $\geqslant 2$. All these brackets are mutually constrained through the set of Jacobi-type identities. The higher Lie structures have applications in Theoretical Physics; see [24] specifically for string theory, and see the recent review [25] (§ 4.1 details a wide range of applications in physics) and references therein for applications. Apart from the mathematical convenience of higher structures (as they serve to resolve obstructions and make proofs work), the idea of N-ary structures with higher arities is that they encode the collective effects beyond the pairwise interactions or beyond the factorisations into only pairwise interactions.

Higher-arity differential structures appear naturally in solid state Physics [3]: specifically, in the integral formula of the topological charge for the (unit) magnetisation vector $\vec{m}(x, y)$ in the plane $\mathbb{R}^2 \ni (x, y)$. The formula of the charge integrates the spherical angle, yielding the number,

$$\frac{1}{4\pi} \iint_{\mathbb{R}^2} \det[\vec{m}, \partial_x \vec{m}, \partial_y \vec{m}]\, dxdy \in \mathbb{Z}, \tag{1}$$

that classifies the mappings $\vec{m}\colon \mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2 \to \mathbb{S}^2 \subsetneq \mathbb{R}^3$; this number is the Brouwer's degree of a map,[1] or, in physics terminology, the topological charge. In effect, the integrand (normalised by the area of the unit sphere) is given by the generalised Wronskian (over the base dimension $d = 2$) for the three Cartesian components of the magnetisation vector $\vec{m} = (m^1, m^2, m^3)$. Its higher-dimensional generalisations, with $\boldsymbol{m}\colon \mathbb{R}^d \cup \{\infty\} \simeq \mathbb{S}^d \to \mathbb{S}^d \subsetneq \mathbb{R}^{d+1}$, involve the complete generalised Wronskians $W^{k=1}_{d\geqslant 2}$ of differential order $k = 1$ over the d-dimensional bases $\mathbb{R}^d$ (under the assumption $\boldsymbol{m}(\infty) = \boldsymbol{m}_0$ about asymptotic behaviour of the vector $\boldsymbol{m}$ on $\mathbb{R}^d$). These (anti-)instanton solutions of field models are explained in much detail in [18] and [20], as well as in [22]. Skyrmions [23] were an attempt to apply this magnetic-field picture to (higher-dimensional) models of the strong interaction, namely for the baryons made conjecturally of pions. The formalism suits well to describe the skyrmion crystals with large topological numbers, see [16] and [9], where polynomials and rational functions are used in the constructions.

The generalised Wronskian determinants for functions in many variables were rediscovered many times: e.g., in Number Theory [15, 21] or in the engineering sciences [26]; cf. [6] for a review of Wronskians in algebraic geometry and related fields, in particular, see [17] for Wronskians connecting number theory and physics through the Andrews–Gordon q-series. Showing up not only in the above formula of topological charge, the generalised Wronskians are immediately useful to detect the linear (in)dependence of differentiable functions; these

[1] It is exactly the integral, $\int_{\mathbb{S}^2} \vec{m}^* \omega$, of the pull-back of the area form ω of the 2-sphere by the map $\vec{m}$ expressed in coordinates (x, y) on the plane with point at infinity $\mathbb{R}^2 \cup \{\infty\}$. See § 1.2 for further details.

determinants appear also in the proof of Roth's theorem about approximations of numbers (see [15, 21]).

Over the base dimension $d = 1$, the usual Wronskian determinants of functions in one variable realise a class of homotopy deformations for Lie algebras of vector fields on the line $\mathbb{R}^1$; this was noted by Dzumadil'daev [4, 5]. The mechanism of those deformations relies on the most basic facts from the theory of associative algebras [8]. In [12], Kiselev extended Dzhumadil'daev's proof to arbitrary dimension $d \geqslant 1$, thus providing a large class of (strongly) homotopy Lie algebra structures on the spaces of functions in d variables; the complete generalised Wronskians $W_{d\geqslant 1}^{k\geqslant 1}$ of (highest appearing) differential order k are the multi-linear anti-symmetric brackets satisfying the table of Jacobi-type identities. In the recent paper [14], Kiselev furthered this result to the case of *in*complete generalised Wronskians of differential orders $k > 1$, thus finally allowing the bracket's arity N to increase incrementally, with the minimal step $+1$ instead of the leaps between the binomial coefficients $N = \binom{d+k}{k}$ as it was in the earlier work [12] (see also [13]).

1.2. **Skyrmions and Wronskians.** Let us examine the construction of a skyrmion field mathematically. Let $\vec{m}(x, y)$ denote the unit magnetisation at each point of the plane; and impose that the magnetisation is local: $\vec{m}$ points up (in $\mathbb{R}^3$) at infinity, that is, $\vec{m}(\infty) = (0, 0, 1)$. Then the plane $\mathbb{R}^2$ can be compactified into the sphere $\mathbb{S}^2$ by using stereographic coordinates $\mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2$. As $\vec{m}$ maps each point (x, y) to a unit vector in $\mathbb{R}^3$, let us write $\vec{m}$ as the map $\vec{m}\colon \mathbb{R}^2 \cup \{\infty\} \simeq \mathbb{S}^2 \to \mathbb{S}^2 \subsetneq \mathbb{R}^3$. To avoid confusion, let $M\colon \mathbb{S}^2 \to \mathbb{S}^2$ correspond to $\vec{m}$. Taking the push-forward into the 2nd homology of the sphere, we obtain $M_*\colon H_2(\mathbb{S}^2) \to H_2(\mathbb{S}^2)$. As $H_2(\mathbb{S}^2; \mathbb{Z}) \cong \mathbb{Z}$ and M_* is a group homomorphism, it follows that M_* must be multiplication by an integer, denoted by Q: $M_* = Q\,\mathrm{id}_{H_2(\mathbb{S}^2)}$. This integer Q is called the Brouwer's degree; for Skyrmions, it is called the topological charge.

Let us calculate Q via integration. The 2nd cohomology (de Rham) of the sphere is also one-dimensional $H^2(\mathbb{S}^2; \mathbb{R}) \cong \mathbb{R}$, and has generator $[\omega]$ that satisfies $\int_{\mathbb{S}^2} \omega = 1$. Putting $\mathbb{S}^2 \subsetneq \mathbb{R}^3 \ni (x, y, z)$, one obtains[2] $\omega = \frac{1}{4\pi}(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy)$. By the definition of the Riemann integral in terms of pull-backs and push-forwards, we have the identity

$$\int_{\mathbb{S}^2} M^*\omega = \int_{M_*\mathbb{S}^2} \omega = Q \int_{\mathbb{S}^2} \omega = Q$$

as ω is the *normalised* area form. Hence the topological charge is the integral of the pull-back of the area form along the smooth map M.

Denote by F the inverse stereographic projection[3] $F\colon \mathbb{R}^2 \cup \{\infty\} \to \mathbb{S}^2$. By definition and properties of the pull-backs of differential forms,

$$Q = \int_{\mathbb{S}^2} M^*\omega = \int_{F(\mathbb{R}^2)} M^*\omega = \int_{\mathbb{R}^2} F^*M^*\omega = \int_{\mathbb{R}^2} (M \circ F)^*\omega = \int_{\mathbb{R}^2} \vec{m}^*\omega,$$

as $\vec{m}(u, v) = M(F(u, v))$ by construction. Let us calculate the pull-back $\vec{m}^*\omega$; by definition, the pull-back along $\vec{m} = (m_1, m_2, m_3)$ at a point with coordinates $(u, v) \in \mathbb{R}^2$ evaluated on

[2]Indeed, $d\omega = \frac{3}{4\pi}\,dx \wedge dy \wedge dz$, hence $\int_{\mathbb{S}^2} \omega = \int_{x^2+y^2+z^2<1} d\omega = \frac{3}{4\pi} \cdot \frac{4}{3}\pi 1^2 = 1$ by Stokes' theorem.

[3]Explicitly, F is given by $F(u, v) = (2u, 2v, r^2 - 1)/(r^2 + 1)$, where $r^2 = u^2 + v^2$. The exact expression for F does not explicitly appear in this derivation.

the two basis (tangent) vectors $\vec{e}_1 = \partial/\partial u$ and $\vec{e}_2 = \partial/\partial v$ of the plane $\mathbb{R}^2 \ni (u, v)$ is

$$\begin{aligned}
\vec{m}^*\omega(u,v)\big(\tfrac{\partial}{\partial u} \wedge \tfrac{\partial}{\partial v}\big) &= \omega(\vec{m}(u,v))(\partial_u\vec{m} \wedge \partial_v\vec{m}) \\
&= \tfrac{1}{4\pi}\big(m_1\, dy \wedge dz\,(\partial_u\vec{m} \wedge \partial_v\vec{m}) - m_2\, dx \wedge dz\,(\partial_u\vec{m} \wedge \partial_v\vec{m}) + \\
&\qquad\qquad + m_3\, dx \wedge dy\,(\partial_u\vec{m} \wedge \partial_v\vec{m})\big) \\
&= \frac{1}{4\pi}\left(m_1 \begin{vmatrix} \partial_u m_2 & \partial_u m_3 \\ \partial_v m_2 & \partial_v m_3 \end{vmatrix} - m_2 \begin{vmatrix} \partial_u m_1 & \partial_u m_3 \\ \partial_v m_1 & \partial_v m_3 \end{vmatrix} + m_3 \begin{vmatrix} \partial_u m_1 & \partial_u m_2 \\ \partial_v m_1 & \partial_v m_2 \end{vmatrix}\right) \\
&= \frac{1}{4\pi} \det \begin{vmatrix} m_1 & m_2 & m_3 \\ \partial_u m_1 & \partial_u m_2 & \partial_u m_3 \\ \partial_v m_1 & \partial_v m_2 & \partial_v m_3 \end{vmatrix} (u,v) \\
&= \tfrac{1}{4\pi} W_{d=2}^{k=1}(m_1, m_2, m_3)(u,v) \\
&= \tfrac{1}{4\pi} W_{d=2}^{k=1}(\vec{m})(u,v),
\end{aligned}$$

i.e. we recover the Wronskian determinant of order one over $\mathbb{R}^2 \ni (u,v)$: $W_{d=2}^{k=1} = \mathbf{1} \wedge \partial_u \wedge \partial_v$. Therefore, the expression for the topological charge becomes

$$Q = \int_{\mathbb{S}^2} M^*\omega = \int_{\mathbb{R}^2} \vec{m}^*\omega = \frac{1}{4\pi} \int_{\mathbb{R}^2} W_{d=2}^{k=1}(\vec{m})\, dxdy,$$

as was claimed in § 1.1 of this Overview. Indeed, the determinant is exactly the triple product $\vec{m} \cdot (\partial_u\vec{m} \times \partial_v\vec{m})$; the intuitive explanation here is that the spherical angle is being integrated. The integrals do not diverge by construction of $\vec{m}$ from $M\colon \mathbb{S}^2 \to \mathbb{S}^2$.

The constructions are also valid in d dimensions as the homologies and cohomologies of the d-sphere $\mathbb{S}^d = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : ||\boldsymbol{x}|| = 1\}$ are $H_d(\mathbb{S}^d;\mathbb{Z}) \cong \mathbb{Z}$ and $H^d(\mathbb{S}^d;\mathbb{R}) \cong \mathbb{R}$. Here, the d-area form is the contraction of the Euclidean $(d+1)$-volume form $\Omega = dx^1 \wedge \cdots \wedge dx^{d+1}$ on $\mathbb{R}^{d+1} \ni (x^1, ..., x^{d+1})$ by the normal vector of the sphere, obtaining

$$\omega = \frac{1}{\operatorname{Area}\mathbb{S}^d} \sum_{i=1}^{d+1} (-)^{i-1} x^i\, dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{d+1}.$$

Again, we denote the inverse stereographic projection by $F\colon \mathbb{R}^d \cup \{\infty\} \to \mathbb{S}^d$, whereby the topological charge is $Q = \int_{\mathbb{S}^d} M^*\omega$, where $M\colon \mathbb{S}^d \to \mathbb{S}^d$ is a smooth map and the corresponding $\boldsymbol{m} = M \circ F = (m_1, ..., m_{d+1})$ is the magnetisation field over $\mathbb{R}^d \cup \{\infty\}$. Denote the coordinates on $\mathbb{R}^d \cup \{\infty\}$ by $\boldsymbol{u} = (u^1, ..., u^d)$, and put, by definition, the gradient vector $\partial m_i/\partial \boldsymbol{u} = [\partial m_i/\partial u^1, ..., \partial m_i/\partial u^d]$. Then the pull-back $\boldsymbol{m}^*\omega(\boldsymbol{u})\big(\frac{\partial}{\partial u^1} \wedge \cdots \wedge \frac{\partial}{\partial u^d}\big)$ of the normalised area form ω along the smooth map $\boldsymbol{m}$ at point $\boldsymbol{u}$, and evaluated on the d-blade $\partial/\partial u^1 \wedge \cdots \wedge \partial/\partial u^d$ of the compactified d-space $\mathbb{R}^d \cup \{\infty\}$, is given by

$$\frac{1}{\operatorname{Area}\mathbb{S}^d} \sum_{i=1}^{d+1} (-)^{i-1} m_i(\boldsymbol{u}) \det\left[\frac{\partial m_1}{\partial \boldsymbol{u}}, ..., \widehat{\frac{\partial m_i}{\partial \boldsymbol{u}}}, ..., \frac{\partial m_{d+1}}{\partial \boldsymbol{u}}\right],$$

where the sum is the determinant

$$\det \begin{bmatrix} m_1 & \cdots & m_{d+1} \\ \partial m_1/\partial u^1 & & \partial m_{d+1}/\partial u^1 \\ \vdots & \ddots & \vdots \\ \partial m_1/\partial u^d & \cdots & \partial m_{d+1}/\partial u^d \end{bmatrix} = W_d^{k=1}(m_1, ..., m_{d+1})(\boldsymbol{u}) = W_d^{k=1}(\boldsymbol{m})(\boldsymbol{u}),$$

now the Wronskian of first order over $\mathbb{R}^d$. This again gives the equivalent expressions for the topological charge

$$Q = \int_{\mathbb{S}^d} M^*\omega = \int_{\mathbb{R}^d} \boldsymbol{m}^*\omega = \frac{1}{\operatorname{Area}\mathbb{S}^d} \int_{\mathbb{R}^d} W_d^{k=1}(\boldsymbol{m})(\boldsymbol{u})\,d\boldsymbol{u},$$

i.e. the normalised integral of the Wronskian determinant of the coordinate functions. Note that in the 3-dimensional case, to each point in 3-dimensional Euclidean space we assign a unit vector with *four* instead of three components. The assignment of a 3-vector does not proceed via this construction.

Let us examine another context in which Wronskians naturally appear. Recall that the Lie bracket of two vector fields $\vec{\mathcal{X}} = \mathcal{X}(x)\,\partial/\partial x$ and $\vec{\mathcal{Y}}$ on the real line $\mathbb{R}^1 \ni x$ is given by $[\vec{\mathcal{X}}, \vec{\mathcal{Y}}] = \vec{\mathcal{X}} \circ \vec{\mathcal{Y}} - \vec{\mathcal{Y}} \circ \vec{\mathcal{X}}$. The expression is again a vector field given by

$$[\vec{\mathcal{X}}, \vec{\mathcal{Y}}] = \begin{vmatrix} \mathcal{X} & \mathcal{Y} \\ \mathcal{X}' & \mathcal{Y}' \end{vmatrix} \frac{\partial}{\partial x} = W^{0,1}(\mathcal{X}, \mathcal{Y})\,\partial/\partial x,$$

where $W^{0,1}$ is the Wronskian determinant of arity two. In particular, $\mathfrak{sl}(2)$ is realised by the three vector fields $\vec{e} = 1\,\partial/\partial x$, $\vec{f} = -x^2\,\partial/\partial x$, and $\vec{h} = -2x\,\partial/\partial x$ with the Lie bracket above: $[\vec{e}, \vec{f}] = \vec{h}$, $[\vec{h}, \vec{f}] = -2\vec{f}$, and $[\vec{h}, \vec{e}] = 2\vec{e}$; cf. [12, 13, ĶK26a].

Let us alternate higher order differential operators $\mathcal{W}_j = w_j(x)\,\partial^p/\partial x^p$ on the real line $\mathbb{R}^d \ni x$, with $j = 1, \ldots, 2p$. Put $N = 2p$. In [12] and [13] it is shown that their alternator becomes

$$\operatorname{Alt}(\mathcal{W}_1, \ldots, \mathcal{W}_N) := \sum_{\sigma \in S_N} (-)^\sigma (\mathcal{W}_{\sigma(1)} \circ \cdots \circ \mathcal{W}_{\sigma(N)}) = c(p) \cdot \underbrace{\begin{vmatrix} w_1 & \cdots & w_N \\ w_1' & \cdots & w_N' \\ \vdots & & \vdots \\ w_1^{(N-1)} & \cdots & w_N^{(N-1)} \end{vmatrix}}_{= W^{0,1,\ldots,N-1}(w_1, \ldots, w_N)} \frac{\partial^p}{\partial x^p},$$

where the constants $c(p)$ are universal, $c(1) = 1$, and the coefficient is the Wronskian of order $N-1$ over $\mathbb{R}^1$ of the original arguments' coefficients $W^{0,1,\ldots,N-1}(w_1, \ldots, w_N)$.

1.3. **Local embedding.** The mathematical aspects of this work embed within two research groups at the Benoulli Institute for Mathematics, Computer Science, and Artificial Intelligence (BI) of the University of Groningen (UG). In particular, firstly, into the Algebra research group, which takes part in the Dutch Research Council (NWO) clusters DIAMANT – for algebra and number theory – and Geometry & Quantum Theory (GQT). Secondly, the geometrical aspects of this work connect to the Dynamical Systems, Geometry & Mathematical Physics (DSGMP) research group; specifically, research on algebraic topology and mathematical physics (theoretical mechanics and quantum mechanical aspects). Within the BI, research on deformation theory (part of Algebra) and Lie structures is primarily conducted by the mathematics advisor Dr A.V. Kiselev.

The physical aspects embed within the van Swinderen Institute for Particle Physics and Gravity (VSI) and the Zernike Institute for Advanced Materials (ZIAM): in VSI, research is conducted on string theory, quark and lepton symmetries, and non-Abelian gauge theories (in varying spacetime dimensions) – in connection with cosmology and condensed matter –; in ZIAM, condensed matter research is extremely broad and active – it is primarily experimental, but also theoretical. Within the Theory of Condensed Matter (ToCM) group of the ZIAM, research on (periodic) spin textures and topological materials and structures (chiefly, the Skyrmions) is very active at present and has been ongoing for decades – it is now well-established. The principal researcher on Skyrmions is the physics advisor Prof. M.V. Mostovoy.

Many previous projects at the UG – on deformation theory, algebraic and cohomological structures in mathematical physics, and the Kontsevich graph calculus – have been supervised by Dr Kiselev over the period from ca 2011 onwards, and at other institutions before that.

1.4. **Research problem.** Back in the purely Lie case of binary brackets over base dimension $d = 1$, Kirillov asked: *given finitely many generic vector fields on the line* $\mathbb{R}^1$*, how fast does the Lie algebra grow under the iteration of commutators?* The answer to this question, given in [10, 11], led to the notion of (Lie) algebras of intermediate growth. The now-available picture of (in-)complete generalised Wronskians over $\mathbb{R}^{d\geqslant 1}$ makes natural the same question in the larger setting: *how fast does the (polynomial) homotopy Lie algebra grow under iteration of the Wronskian for arguments in* d *independent variables?*

1.4.1. *Problem about finite dimension.* The reference point along this line of research is: *to describe all the polynomial algebras that remain finite-dimensional, i.e. do not grow at all.*

We solve this problem in [ĶK26a] (§ 2 of this thesis).

1.4.2. *Problem about polynomial degree growth.* Conversely, when the dimension of a finitely-generated Lie algebra is *not* bounded, *the task is to control – at least asymptotically – the polynomial degrees (w.r.t.* d *independent variables) and the dimension of the subspaces which are spanned within the algebra after finitely many iterations of the Wronskian bracket.*

We solve the first part of this problem in [ĶK26b] (§ 3 of this thesis).

1.4.3. *Research subject.* This research is interdisciplinary within mathematics and physics (field theories and topological solid-state); the slogan for the constructions, objects, and applications is "*higher structures in physics*". It refers to concepts and is conducted within the fields such as: • Lie algebras, deformations of Lie algebras, (strongly) homotopy Lie algebras, N-ary Lie algebras, growth of Lie algebras (under iteration, in degree or size), growth of homotopy Lie algebras, realisation theory, asymptotics of integer recurrences, factorisations of generalised Vandermonde determinants; • solid-state physics, topological condensed matter physics and materials, topological invariants, homology of spheres, homotopy classes, differential constructions in algebraic topology, Skyrmions (field configurations, magnetic), topological stability of fields, lowest energy topological fields; • higher structures in physics (homotopy Lie and differential), N-point interactions, gauge field theories, conformal field theory, string theory, mathematics for next-generation models in gauge field theory.

1.4.4. *Research object.* Vector fields (on manifolds), jet spaces, vector fields with polynomial coefficients, alternated compositions (in associative algebras), Wronskian determinants and their generalisations over $\mathbb{R}^d$, Vandermonde determinants and their generalisations over $\mathbb{R}^d$, N-ary brackets, Schlessinger–Stasheff identities, Lie algebra $\mathfrak{sl}(2)$ and its deformations, Witt algebra and its deformations and generalisations over $\mathbb{R}^d$, Brouwer's degree (equivalently, called winding number, topological charge, baryon number) and its relation to the strong interaction, stereographic projection onto compactified Euclidean spaces, Cartesian coordinate expressions of topological constructions, realisations of topological constructions by Wronskian determinants.

1.4.5. *Research strategy.* The strategy depends on the problem, as explained below.
• *For Problem 1.4.1.* We use very basic properties about monomials and polynomials, Lie algebras, and determinants to work out by hand simple examples which sit as the most basic cases in our constructions. We write and use computer algebra software to work out more examples and arrive at conjectures about the form of general identities and facts. We then use basic properties to prove the conjectured identities; once proven, they are used to prove general facts (now theorems) and solve problems.

At each point along the process, we examine whether our identities and constructions pattern upon previously known constructions; if so, we formulate our results as extensions (deformations, analogues, generalisations) of previous results, and examine, whether the more general results explain the mechanisms underlying the previous results.
• *For Problem 1.4.2.* We use basic facts about iterating Lie brackets: all choices to select arguments – which ones give the limiting behaviour? Here we find the mechanisms for the degree of monomials to grow, producing integer sequences – we observe whether recurrent behaviour occurs. If so, we then use basic properties of integer recurrences and their asymptotics. We find upper bounds for degree growth and attain them.
• *For connections to Skyrmions.* We examine the constructions used by physicists, looking for differential operators, among which chiefly the Wronskian determinants. We then examine the origin of such structures by starting from general coordinate-free constructions in algebraic topology in order to reveal the mechanism how the differential structures arise. We examine the various methods of computations of quantities, linking them, and explicitly performing calculations according to the simplest method. Again, we view the structures in conjunction with previous work, looking for patterns and deformations of structures; once found, we put all structures in a unified context according to the language and basic principles of deformation theory.

1.5. **Scientific novelty of results.** The main results, which reflect the scientific novelty and which are presented for defence are these:

(1) All finite-dimensional polynomial N-ary Lie algebras $\Bbbk_k[x^1,\ldots,x^d] \subsetneq \mathcal{A} \subsetneq \Bbbk[x^1,\ldots,x^d]$, for the base field choices $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, with the complete generalised Wronskian determinants – of differential order $k \geqslant 1$ over $\mathbb{R}^d$ – as N-ary brackets, $N = \binom{d+k}{k}$, are $\mathcal{A} = \Bbbk_k[x^1,\ldots,x^d] \oplus \langle p(\boldsymbol{x})\rangle$ for any homogeneous polynomial p of degree $k+1$; their dimension is $\dim \mathcal{A} = N + 1 < \infty$. (See Theorems 14 and 17 on pp. 38 and 50,[4] respectively, in § 2 of this thesis, that is, [ĶK26a].)
(2) The structure constants of the class of finite-dimensional algebras are found explicitly and depend only on the polynomial $p(\boldsymbol{x})$. (See Theorem 13 on p. 37 in § 2.)
(3) The complete generalised Wronskian determinant of monomial arguments is again a monomial, whose degree is calculated explicitly, and whose overall coefficient is the complete generalised Vandermonde determinant of the monomial arguments' degrees. (See Theorem 7 on p. 30 in § 2.)

[4]The page numbers refer to the pages in this thesis, that is, not the pages as enumerated in the `arXiv` version.

(4) The correspondence above (pt. № 3) allows determination of non-trivial vanishing conditions of the complete generalised Vandermonde determinant (see Proposition 9 on p. 32 in § 2). It also allows for factorisation of these Vandermonde determinants for certain choices of arguments – see Eqs (37), (52), and (58a–d) on pp. 37, 44, and 48, resp., in § 2.

(5) We find a generalisation of the Witt algebra: from over $\mathbb{C}$ to over $\mathbb{C}^d$ and higher bracket arity N. The elements of the algebra are indexed by multi-indices in $\mathbb{Q}^d$; the multi-indices are added component-wise. The N-ary bracket – complete generalised Wronskian – of N indexed arguments is the element, whose multi-index is the component-wise sum of the arguments' multi-indices; the structure constants are totally antisymmetric in the multi-indices. (See Remark 8 on p. 32 in § 2.)

(6) We establish an asymptotic upper bound for the growth in degree of polynomials under iteration of the N-ary bracket given by the complete generalised Wronskian determinant of differential order k over $\mathbb{R}^d$, $N = \binom{d+k}{k}$. That is, take any finite-dimensional subspace of polynomials over $\mathbb{R}^d$; put the polynomials as arguments of the Wronskian, obtaining possibly new polynomials: add these as generations of the subspace, and repeat inductively. The growth *in degrees* of these polynomials is asymptotically bounded above by the growth of the N-bonacci numbers – these grow exponentially. We attain this bound for certain choices of base dimension d and order k. With this bound for the degrees, we also bound the growth in dimension of the algebra upon iteration of the bracket: see the Theorem and Corollary in § 3 of this thesis – [ĶK26b]. (Cf. Open problem 10 in § 1.6 of this Overview.)

1.6. **Open problems.** In the course of this research, several problems were formulated, which, at the time of writing, stand open. They are given below (with references to where we reported the problems if applicable). Please refer, *while reading*, to the formulas and facts referenced in our preprints, where indicated; the page numbers refer to the *pages in the this thesis*. These problems are relevant, interesting, and sufficiently close to the state of the field at present, to be solvable within a semester; as such, they are good research problems for bachelor's, master's, and parts of PhD theses.

(1) The finite-dimensional polynomial N-ary Lie algebras – with complete generalised Wronskians over $\mathbb{R}^d$ of differential order k as N-ary brackets, $N = \binom{d+k}{k}$ – we constructed are of the form $\mathcal{A} = \Bbbk_k[x^1, ..., x^d] \oplus \langle p(\boldsymbol{x}) \rangle$ for any homogeneous polynomial p of degree $k+1$. Over any base dimension d and any order k *does there exist an eligible polynomial* p *such that the algebra is perfect*, i.e. $\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, ..., \mathcal{A}) = \mathcal{A}$? See Open problem 1 and commentary on p. 40 in § 2 of this thesis, that is, [ĶK26a].

(2) In [ĶK26a], all generalised Wronskians were complete – there are no missing differential operators up to order k. Suppose the generalised Wronskian is incomplete; e.g., over $\mathbb{R}^2 \ni (x, y)$ of order $k = 2$ the top-order differential operator $\partial^2/\partial x \partial y$ is missing from the Wronskian $\mathbf{1} \wedge \partial_x \wedge \partial_y \wedge \partial^2_{xx} \wedge \partial^2_{yy}$ of arity $N = \binom{2+2}{2} - 1 = 5$. *What are the finite-dimensional polynomial* N*-ary Lie algebras with the incomplete Wronskians as* N*-ary Lie brackets?* (See [14] for further information about incomplete Wronskians.)

(3) In [ĶK26a], our main formula, the result of a Wronskian determinant of certain monomial arguments – its r.-h.s. replicated below,

$$\frac{1}{k!\,(k-1)!}\prod_{\ell=2}^{k}(\deg p-\ell)(\deg q-\ell)\cdot\{(\deg p-1)m_1-(\deg q-1)n_1\}\frac{pq}{x^1}$$

(see Theorem 16 and Eq. (52) on p. 44 in § 2) –, vanishes for specific choices of the two free arguments $p=(x^1)^{n_1}\cdots(x^d)^{n_d}$ and $q=(x^1)^{m_1}\cdots(x^d)^{m_d}$: namely, when $\deg p=\deg q$ and $\deg_{x^1}(p)=\deg_{x^1}(q)$. *What is the algebraic mechanism for the Wronskian determinant to vanish in this case?* See Problem 2 on p. 44 in § 2.

(4) The structure constants for our finite-dimensional N-ary Lie algebras can be written as a differential operator acting on the $(k+1)$th degree argument – see Eq. (38) on p. 37 in § 2. The Main formula (52) and more general formulas (58a–d) on pp. 44 and 48 in § 2, resp., are written as a product of two factors: the first is symmetric in the two free arguments, whereas the second is antisymmetric. These formulas pattern upon the formula (38) for the structure constants, where the second term is a differential operator in one free argument.[5] The second terms in (52) and (58) are, respectively,

$$\begin{aligned}\{p,q\}_{\widehat{x}}&=\{(\deg p-1)m_1-(\deg q-1)n_1\}\frac{pq}{x},\\ \{p,q\}_{\widehat{(x)^2}}&=\{(\deg p-1)(\deg p-2)m_1(m_1-1)-\\ &\qquad-(\deg q-1)(\deg q-2)n_1(n_1-1)\}\frac{pq}{(x)^2},\end{aligned}\tag{$*$}$$

where $x=x^1$, $p=(x^1)^{n_1}\cdots(x^d)^{n_d}$, and $q=(x^1)^{m_1}\cdots(x^d)^{m_d}$ (see Eqs (56) and (57) on p. 48 in § 2). *Can the formulas in ($*$) be written as antisymmetric differential operators acting on the free monomials p and q?* See Open problem 3 on p. 49 in § 2.

(5) It is known that the (complete) generalised Vandermonde determinants do not factorise into a product of linear (or higher degree) combinations of the arguments; see [2], where the determinants are expanded into a sum of smaller Vandermonde determinants, which do factorise. Nevertheless, in simple cases factorisation is possible: see Eqs (37), (52), and (58a–d) on pp. 37, 44, and 48, resp., in § 2. In these cases most arguments for the Wronskian determinant are chosen from monomials up to degree k, whereby the few free (monomial) arguments replace specific monomials as arguments; for instance, put as the N arguments all monomials up to degree k except 1 and x^1 – in their place put arbitrary monomials p and q. The Wronskian determinant of these arguments is a monomial with the generalised Vandermonde determinant of the monomials' degrees as the constant (see Theorem 7 in § 2): *can the complete generalised Vandermonde determinant be factorised into a product of terms?* Do this for specific choices of monomials to replace by free monomials, for instance, 1 and xy, or a higher number of free arguments – for instance, replace 1, x, and y by three free monomial arguments p, q, and r. See Open problem 4 on p. 49 in § 2.

[5] If there are more than two free arguments, it is conjectured that the formulas will likewise split into symmetric and anti-symmetric terms, where the second is totally antisymmetric w.r.t. all free arguments; indeed, the Wronskian and Vandermonde determinants are totally antisymmetric.

(6) In our two Main Theorems (14 and 17 on pp. 38 and 50, resp., in § 2), we demanded that the polynomial N-ary Lie algebras must contain all monomials up to degree k as generators: $\mathcal{A} \supsetneq \Bbbk_k[x^1, ..., x^d]$. If we relax this assumption, we also get many 'trivial' N-ary Lie algebras: (*i*) of dimension N, e.g., $\mathcal{A} = \langle x, y, y^3 \rangle$ with $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$ as ternary bracket, or (*ii*) that do not span all directions in the base $\mathbb{R}^d$, e.g., $\mathcal{A} = \langle 1, y, y^{10}, ..., y^{100} \rangle$ with $W^{k=1}_{d=2}$ as ternary bracket over base $\mathbb{R}^2 \ni (x, y)$ – here all choices of arguments evaluate under the Wronskian to zero. A way to remedy these trivialities is to demand that the algebra $\mathcal{A}$ be perfect. *Do there exist non-trivial (perfect?) polynomial finite-dimensional N-ary Lie algebras that do not contain all polynomials up to degree k: $\Bbbk_k[x^1, ..., x^d] \nsubseteq \mathcal{A}$?* See Open problem 5 on p. 52 in § 2.

(7) In [ĶK26a], Remark 8 on p. 32 (cf. Remark 2 on p. 25), we wrote a generalisation of the Witt algebra (Virasoro algebra from CFT with zero central charge) from over base $\mathbb{C}$ to over higher complex-dimensional base $\mathbb{C}^d$. The strong homotopy deformations (over $\mathbb{C}$) of the Witt algebra – to higher bracket arities – were found by Dr Kiselev in [13]. *What are the applications (in physics) of these higher-arity and higher-base-dimension generalisations of the Witt algebra?*

(8) The Brouwer's degree (topological charge) of a continuous mapping between two d-spheres, $\boldsymbol{m} \colon \mathbb{S}^d \to \mathbb{S}^d$, can be calculated using stereographic coordinates on the domain unit sphere (see §§ 1.1 and 1.2 of this Overview), where the structure integrated is the generalised Wronskian determinant of first order over $\mathbb{R}^d$. We ask: *are there topological invariants or constructions, where higher order Wronskian determinants appear?* If so, *what are the applications to (topological) condensed matter physics?*

(9) Fix the base dimension d and differential order k of the Wronskian W^k_d with arity $N = \binom{d+k}{k}$. To every argument add the result of some differential operator P acting on it: $W^k_d(f + P(f), g + P(g), ...)$. Take some finite-dimensional N-ary Lie algebra $\mathcal{A}$ from our class: $\mathcal{A} = \Bbbk_k[x^1, ..., x^d] \oplus \langle p(\boldsymbol{x}) \rangle$, $\deg(p) = k+1$. *For what differential operators P are the commutation relations of the algebra $\mathcal{A}$ preserved?* I.e. for all choices of arguments $f, g, ... \in \mathcal{A}$ we have $W^k_d(f + P(f), g + P(g), ...) = W^k_d(f, g, ...)$.

These operators P are often called *gauge symmetries*.[6] Note that differential operators, which vanish on the entire algebra, $P(f) = 0$ for every $f \in \mathcal{A}$, are trivially gauge symmetries; these are, for instance, homogeneous differential operators, whose degree exceeds $k + 1$ (e.g., $\partial^3/\partial x^3$ over $\mathcal{A} = \langle 1, x, y, xy \rangle$). We are interested specifically in non-trivial (if they exist) gauge symmetries.

(10) (*Kirillov problem extended to the N-ary case*)**.** Let $\mathcal{A}$ be an N-ary Lie algebra with bracket $[\cdot, \cdots, \cdot]_N$. Take any finite-dimensional subspace $V \subseteq \mathcal{A}$ and insert the elements of V as arguments of the bracket. As outputs, possibly new (not in V) elements of $\mathcal{A}$ are created, denote their span by $\langle [V, \cdots, V]_N \rangle$; append these as generators of a larger subspace $V_1 = V \oplus \langle [V, \cdots, V]_N \rangle$. Repeat this process with V_1 in place of V, obtaining $V_2, V_3, ...$. If new elements are created at every step, the dimensions of the sequence of subspaces $V, V_1, V_2, ...$ will grow. *Among all choices for the subspace V, what is the fastest possible growth in dimension upon iterating the N-ary bracket?*[7]

[6] Note that operators of totally different form on differential structures are also called gauge transformations.

[7] See [7] for the notion of an asymptotic dimension in the case of Lie algebras. When this growth is slower than exponential in the number of iterations, but the algebras grow faster than polynomials, the algebras are called of intermediate growth – see [10] and [11].

We specifically ask this question for polynomial N-ary Lie algebras with the complete generalised Wronskian determinants as N-ary brackets. (Note that we have answered the case when this dimension does not grow at all – the algebra $\mathcal{A}$ is finite-dimensional.)

1.7. **Practical significance of results.** In this thesis, we develop mathematical tools for the construction of next-generation models in gauge field theory, in particular, for collective effects. Purely mathematically, the results are the reference point of a much larger problem about the growth of (homotopy) Lie algebras. As the case of purely Lie algebras led to interesting classes of Lie algebras (of intermediate growth, see [11]), it is very possible that just as interesting notions can be found in the case of N-ary Lie algebras, or homotopy Lie algebras in general.

We connect our results very naturally to combinatorics and number theory: integer recurrences, their asymptotics, and generalised Fibonacci numbers. In addition, to a very recent line of research by Prof. F. Brown about generalised Vandermonde determinants [2], yet we attack his problem from the totally opposite direction. Because of this, the study of polynomial N-ary Lie algebras with Wronskians as brackets serves as a natural source of connections to be found between these various fields of mathematics, with applications to different branches of physics. The web of connections is shown in Figure 1.

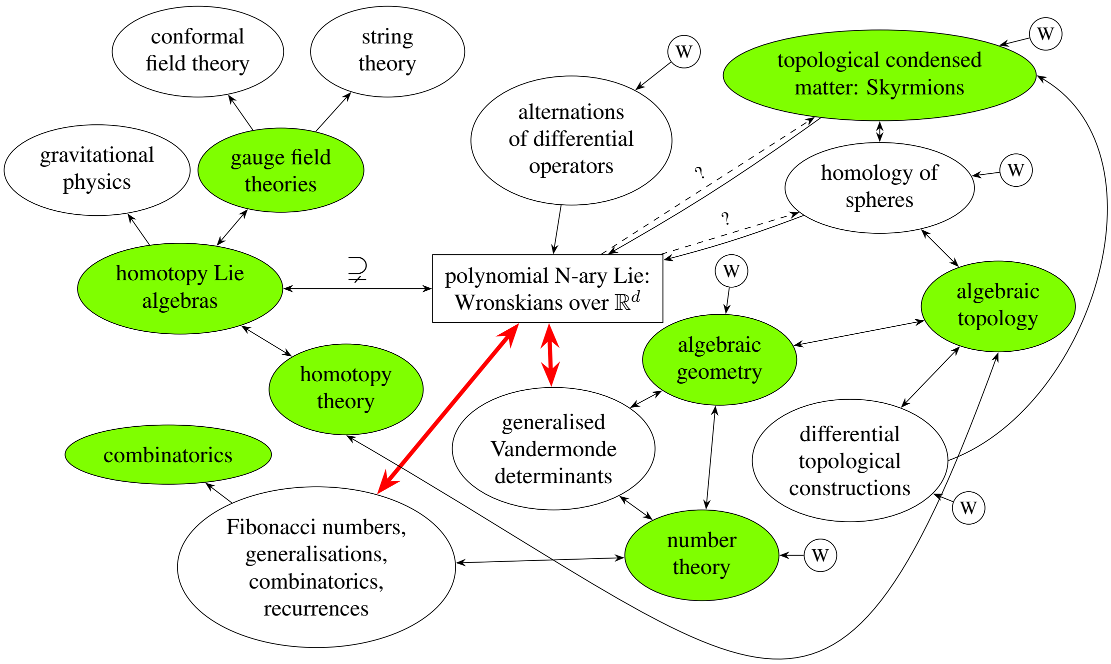

FIGURE 1. Relation of polynomial N-ary Lie algebras – with generalised Wronskian brackets – as higher structures in physics to other fields of physics and mathematics (shaded in **green**) and specific theories and constructions within them (not shaded). The symbol 'W' denotes that Wronskian determinants naturally appear in the field (cf. [6]); arrows highlighted in **thick red** indicate that (to the best of our present knowledge) we were the first to find the connection.

1.8. **Personal contribution.** All results, which are presented in this thesis, the graduate accomplished himself. In the jointly-written works, the contributions of the graduate are as follows.

In [ĶK26a], I did all calculations and proofs which are presented as new results (i.e. not replicated from references); the results were frequently discussed with the mathematics advisor Dr Kiselev, who gave valuable advice, left many comments, and made many suggestions for clarification and reformulation. I wrote all sections except the Introduction until Example 4; all writing was done in frequent and extensive consultation with Dr Kiselev, who made many edits and improvements, especially to the early drafts.

In [ĶK26b], I did all the calculations and proofs myself; the results were discussed with Dr Kiselev in great detail. The abstract, introduction, and conclusions were written jointly by myself and Dr Kiselev; I wrote the rest of the first manuscript myself. Several modifications and improvements (towards the `arXiv` version) were later made by Dr Kiselev.

The connections between the Wronskian determinant of first order over two dimensions and the topological charge (Brouwer's degree) were found during a discussion with the physics advisor Prof. Mostovoy. More connections were made and literature sources were compiled during a discussion between both advisors and myself together. Further connections and formulations I noticed myself: either from constructions presented, from personal knowledge, or from reading relevant literature. Experimentation with constructions of Skyrmions of low topological charge I did myself.

1.9. **Approbation of results.** While at IHÉS (May & June 2026), the mathematics advisor Dr Kiselev discussed the work with Prof. M. Kontsevich, who left valuable comments and advice, which have been incorporated into this thesis.

The graduate defended this thesis to both supervisors on the 25th of June, 2026.

The results of this thesis were presented by both the graduate and Dr Kiselev, respectively, at the following events:

1.9.1. *International conference.* The 36th International Colloquium on Group Theoretical Methods in Physics 'Group36' (Valladolid, Spain, July 13–17, 2026).

1.9.2. *Seminar.* Bernoulli Institute Algebra Seminar (Groningen, the Netherlands, June 16; one hour blackboard talk plus 30 min of discussion).

1.9.3. *Talks by the mathematics supervisor.* Dr Kiselev gave two talks about [ĶK26a]: at the Prague Mathematical Physics seminar (Charles University, Czech Republic, May 7, 2026) and the Mathematics seminar (IHÉS, Bures-sur-Yvette, France, May 13, 2026).

1.10. **Structure of the thesis.** This thesis consists of two publications (preprints, one to-be-announced) – about polynomial N-ary Lie algebras – and details about connections to Skyrmions given in § 1.1 and § 1.2 of this Overview.

1.11. **Conclusions.** We have found all finite-dimensional N-ary Lie (Schlessinger–Stasheff strongly homotopy) algebras built from polynomials over $\mathbb{R}^d$ and with the complete generalised Wronskian determinants – of differential order k over $\mathbb{R}^d$ – as N-ary Lie brackets, $N = \binom{d+k}{d}$, that contain no gaps in polynomials up to degree k: $\Bbbk_k[x^1, \ldots, x^d] \subsetneq \mathcal{A} \subsetneq \Bbbk[x^1, \ldots, x^d]$. (Here $\Bbbk = \mathbb{R}$ or $\mathbb{C}$.)

This answers our first research question; it also answers the base case to the much larger problem asked by Kirillov about growth of (in our case, N-ary) Lie algebras upon iteration:[8] we have found algebras that do not grow at all. The study of growth is one of the next steps for future research; here we have reduced the space that can be occupied by the newly-created polynomials upon iteration: asymptotically, their degrees do not exceed the N-bonacci numbers. We attain this bound for certain choices of base dimension and order. This is again the base step in this direction and answers the second research question.

The (generalised) Wronskian determinants appear naturally over $\mathbb{R}^1$ from alternations of differential operators, whereas over $\mathbb{R}^d$ of first order: from topological constructions. It is currently unknown, and is of great interest, how generalised Wronskians of higher order over $\mathbb{R}^d$, $d > 1$, appear naturally in constructions in physics and mathematics; due to geometric connections through the generalised Vandermonde determinants,[9] the question is worth further study.

The results of this thesis are aimed to advance the state of the field. We have formulated the questions that stand immediately beyond our findings; these are highly interesting and relevant open problems. See § 1.6 of this Overview for the full list of Open problems; some of these questions have already been formulated in our preprints.

1.12. **Publications.** The results of this thesis are contained in the following two preprints.

[ĶK26a] Ķēniņš M. G., Kiselev A. V. (2026) Explicit classs of finite-dimensional polynomial algebras with Wronskians over $\mathbb{R}^d$ as N-ary Lie brackets: Beyond $\mathfrak{sl}(2)$. (*Preprint* arXiv:2605.27305) [math.RA, *cross-ref.:* math-ph, math.CO, math.QA], 44 p.

[ĶK26b] Ķēniņš M. G., Kiselev A. V. (2026) Iterate Wronskians over $\mathbb{R}^d$ as N-ary brackets on $\mathbb{R}[x^1, ..., x^d]$: the N-bonacci numbers bound the highest total degrees. (*Preprint* arXiv:2607.26039) [math.RA, *cross-ref.:* math-ph, math.CO, math.QA], 13 p.

References

[1] Alexandrov, M., Kontsevich, M. L., Schwarz, A., and Zaboronsky, O. (1997) The geometry of the master equation and topological quantum field theory. *Int. J. Mod. Phys.* **12** 71405–1430

[2] Brown, F. (2025) Multivariable Vandermonde determinants, amalgams of matrices and Specht modules. *J. Algebra* **678** 253-278 DOI: https://doi.org/10.1016/j.jalgebra.2025.03.040.

[3] Brown, G. E. and Rho, M. (Eds.). (2010) "*The Multifaceted Skyrmion*" World Scientific Publishing Co. Pte. Ltd., Singapore.

[4] Dzhumadil'daev, A. S. (1988) Integral and mod p-cohomologies of the Lie algebra W_1. *Funct. Anal. Appl.* **22** 3 226–228.

[5] Dzhumadil'daev, A. S. (2005) n-Lie structures generated by Wronskians. *Siberian Math. J.* **46** 4 601–612 DOI: https://doi.org/10.1007/s11202-005-0061-7. (*Preprint* arXiv:math/0202043)

[8] See Open problem 10 in § 1.6 of this Overview.

[9] See Theorem 7 on p. 12 of [ĶK26a]; equivalently, main result (3) in § 1.5 of this Overview. Cf. [2].

[6] Gatto, L. and Scherbak, I. (2012) On Generalized Wronskians. *EMS Ser. Congr. Rep.* Proc. 'The Impanga Conference on Algebraic Geometry', in 'Contributions to algebraic geometry: Impanga Lecture Notes', Eur. Math. Soc., Zürich DOI: https://doi.org/10.4171/114-1/9 (*Preprint* `arXiv:1310.4683`)

[7] Gel'fand, I. M. and Kirillov, A. A. (1966) Sur les corps liés aux algèbres enveloppantes des algèbres de Lie (transl. from Russian). *Publications Mathématiques de l'IHÉS* **31** 5–19. Available at URL: http://eudml.org/doc/103872

[8] Hanlon, P., Wachs, M. On Lie k-Algebras, *Adv. Math.* **113** 2 206–236.
DOI: https://doi.org/10.1006/aima.1995.1038

[9] Houghton, C. J., Manton, N S., and Sutcliffe, P. M. (1997) Rational maps, monopoles and Skyrmions. *Nucl. Phys. B* **510** 3 507–537. (*Preprint* `arXiv:hep-th/9705151`)

[10] Kirillov, A. A. and Kontsevich, M. L. (1983) The growth of the Lie algebra generated by two generic vector fields on the line. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, no. 4, 15–20.

[11] Kirillov, A. A., Kontsevich, M. L., and Molev, A. I. (1983) Algebras of intermediate growth. *Akad. Nauk SSSR Inst. Prikl. Mat. Preprint 1983*, no. 39, 19 pp. (Transl. in: *Selecta Math. Soviet.* (1990) **9**:2, 137–153.)

[12] Kiselev, A. V. (2007) On associative Schlessinger–Stasheff algebras and Wronskian determinants. *J. Math. Sci.* **141** 1, 1016–1030
DOI: https://doi.org/10.1007/s10958-007-0028-2. (*Preprint* `arXiv:math/0410185`)

[13] Kiselev, A. V. (2025) Wronskians as N-ary brackets in finite-dimensional analogues of $\mathfrak{sl}(2)$. *J. Phys.: Conf. Ser.* **3152** Proc. XXIX Int. conf. 'Integrable Systems & Quantum Symmetries' (7–11 July 2025, CVUT Prague, Czech Republic), Paper 012044. — 8 p. (*Preprint* `arXiv:2510.02145`)

[14] Kiselev, A. V. (2025) Jacobi identities for Wronskian determinants over multidimension. *Bulgarian J. Phys.* **52**-s1 Quantum Theory & Symmetries, 102–107. (*Preprint* `arXiv:2511.03848`)

[15] LeVeque, W. J. (1956) "*Topics in number theory*". Vol. **2**. Addison–Wesley Publ. Co., Reading MA, pp.128–134.

[16] Manton, N. S. (2012) Classical Skyrmions – static solutions and dynamics. *Math. Meth. Appl. Sci.* **35** 1188–1204. DOI: https://doi.org/10.1002/mma.2512 (*Preprint* `arXiv:1106.1298`)

[17] Milas, A., Mortenson, E, and Ono, K. (2008) Number theoretic properties of Wronskians of Andrews-Gordon series. *Int. J. Number Theory* **4** 2 323–337
DOI: https://doi.org/10.1142/S1793042108001377 (*Preprint* `arXiv:math/0512623`)

[18] Monastyrsky, M. (1993) "*Topology of Gauge Fields and Condensed Matter*". Springer New York, NY DOI: https://doi.org/10.1007/978-1-4899-2403-2

[19] Polyakov, A. M. (1987) "*Gauge Fields and Strings*". Vol. **3** in Contemporary Concepts in Physics, Harwood Academic Publishers GmbH, Chur, Switzerland.

[20] Rajaraman, R. (1982) "*Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*". North–Holland Publishing Co., Amsterdam

[21] Schmidt, W. M. (1980) "*Diophantine approximation*". *Lect. Notes in Math.* vol. **785**. Springer, Berlin, pp.114–150.

[22] Schwarz, A. S. (1994) "*Topology for Physicists*". Springer Berlin, Heidelberg DOI: https://doi.org/10.1007/978-3-662-02998-5

[23] Skyrme, T. H. R. (1961) Particle states of a quantized meson field. *Proc. R. Soc. A* **262** 1309 237–245. DOI: https://doi.org/10.1098/rspa.1961.0115

[24] Stasheff, J. and Lada, T. (1993) Introduction to SH Lie algebras for physicists. *Internat. J. Theoret. Phys.* **32** 7 1087–1103 DOI: https://doi.org/10.1007/BF00671791. (*Preprint* `arXiv:hep-th/9209099`)

[25] Stasheff, J. (2019) L-infinity and A-infinity structures. *Higher Structures* **3** 1 292–326 DOI: https://doi.org/10.21136/HS.2019.07

[26] Wolsson, K. (1989) Linear dependence of a function set of m variables with vanishing generalized Wronskians. *Linear Algebra Appl.* **117**, 73–80 DOI: https://doi.org/10.1016/0024-3795(89)90548-X.

Research articles

In what follows we include the two preprints that constitute this thesis.

1. The first preprint (44 p., § 2 of this thesis) is available on the `arXiv` server, and is concerned with finite-dimensional polynomial N-ary Lie algebras.

2. The second preprint[10] (13 p., § 3 of this thesis) is again available on the `arXiv` server, and establishes an upper bound for growth of dimension upon iterating the bracket, i.e. Kirillov's problem extended to the case of N-ary Lie algebras (the problem was suggested by M. Kontsevich), see also the Gel'fand–Kirillov dimension.

Note that the content of the preprints may change in the future (i.e. next `arXiv` versions), and the first preprint may be split into multiple articles during the publishing process.

[10] Updated on September 1, 2026. The original thesis contained a reduced version of the preprint in preparation.

EXPLICIT CLASS OF FINITE-DIMENSIONAL POLYNOMIAL ALGEBRAS WITH WRONSKIANS OVER $\mathbb{R}^d$ AS N-ARY LIE BRACKETS: BEYOND $\mathfrak{sl}(2)$

MARKUSS G. ĶĒNIŅŠ[*,⋆] AND ARTHEMY V. KISELEV[*,§]

Based on the talks given by the first author at the Algebra seminar (Bernoulli Institute, Groningen, the Netherlands) and by the last author at the Prague Mathematical Physics seminar (Charles University, Czech Republic) and at the Mathematics seminar (IHÉS, Bures-sur-Yvette, France).

ABSTRACT. Lie algebra $\mathfrak{sl}(2)$ can be realised by vector fields on $\mathbb{R}^1 \ni x$ with polynomial coefficients 1, $-2x$, $-x^2$; their Wronskian determinants yield the Lie bracket. Likewise, the monomials $1, \ldots, x^k/k!, \ldots, x^N/N!$ span finite-dimensional strong homotopy (SH) Lie algebras with the Wronskians $\mathbf{1} \wedge \partial_x \wedge \ldots \wedge \partial_x^{N-1}$ as the N-ary brackets. Over dimension $d = 2$ with $\mathbb{R}^2 \ni (x, y)$ and for the complete generalised Wronskian $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$ of differential order $k = 1$ as the ternary bracket, the finite-dimensional polynomial SH-Lie algebras are spanned by $\langle 1, x, y, p\rangle$ with $p \in \{x^2, xy, y^2\}$.

We explicitly describe all finite-dimensional polynomial SH-Lie algebras $\Bbbk_k[\boldsymbol{x}] \subseteq \mathcal{A} \subseteq \Bbbk[x^1, \ldots, x^d]$ (over $\Bbbk = \mathbb{R}$ or $\mathbb{C}$) with the complete generalised Wronskians $W^{k\geqslant 1}_{d\geqslant 1}$ of order k as N-ary brackets: $N = \binom{d+k}{d}$. We obtain a factorisation formula for the generalised Vandermonde determinants which show up in the structure constants of the polynomial algebras $\mathcal{A}$.

Introduction. On the real line $\mathbb{R}^1$, take finitely many vector fields $\vec{\mathfrak{X}}_j \in \mathbb{D}_{p=1}(\mathbb{R}^1)$ with smooth, in particular polynomial, coefficients $X_j(x)$, and generate the Lie algebra by taking the iterated commutators. Kirillov asked: how fast does it grow ? Specifically, what is the dimension of the subspace spanned by all products of generating elements X_j, with number of factors $\leqslant n$. Kontsevich and Molev joined this investigation of Lie algebras of intermediate growth: see [4] and [5].

Suppose now that the Lie algebra of (polynomial) vector fields is deformed, with the new brackets extended by alternating compositions of differential operators of strict order $p \in \mathbb{N}_{\geqslant 1}$, to a strongly homotopy (SH) Lie algebra, see [2] and [6, 7] as well as [12, 13] with more references therein. The multi-linear antisymmetric brackets of even arities $N = 2p$ are expressed by the Wronskian determinants (attaching to the Lie case at $p = 1$ and $N = 2$, see [11]). Dzhumadil'daev showed in [2] that the entire table of strongly homotopy Jacobi identities remains valid for arbitrary arities of the Wronskians as brackets over $\mathbb{R}^1$.

Date: 10 July 2026.

2010 *Mathematics Subject Classification.* 05E15, 15A15, 17A42, 17B65; secondary 05A10, 05A19, 58H15.

Key words and phrases. Determinant identities, differential-polynomial identities, finite-dimensional algebra, iterated commutators, multivariate Vandermonde determinant, multivariate Wronskian determinant, N-ary Lie bracket, polynomial algebra, $\mathfrak{sl}(2)$, Vandermonde determinant factorisation.

[*] *Address*: Bernoulli Institute for Mathematics, Computer Science & Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK Groningen, the Netherlands.

[⋆] *Address for correspondence* (after 1 September 2026): ETH Zürich – Department of Mathematics, Rämistrasse 101, CH-8092 Zürich, Switzerland.

[§] Corresponding author. *E-mail*: `A.V.Kiselev@rug.nl`.

In [6], the second author re-proved this fact for complete generalised Wronskians over multi-dimensional base, $d = \dim \mathbb{R}^d \geqslant 1$, then relaxing in [8] the assumption that either Wronskian be complete in the Jacobi-type identities which are bi-linear w.r.t. the brackets.

Kirillov's question – how fast does the algebra grow under the iteration of Wronskian brackets ? – now applies to the N-ary structures of differential order $k \geqslant 1$ over the base dimension $d \geqslant 1$. (We have that $N = \binom{d+k}{d}$ for the complete generalised Wronskians $W^{k\geqslant 1}_{d\geqslant 1}$.)

To install the reference point along this line of research, in the present paper we classify the polynomial SH-Lie algebras $\mathcal{A} \supseteq \Bbbk_k[x^1, \ldots, x^d]$ which *do not grow* at all but remain finite-dimensional. Surprisingly, the internal structure of these algebras over arbitrary dimension $d \geqslant 1$ patterns upon the N-ary generalisations $\bigl(\Bbbk_N[x], W^{0,1,\ldots,N-1}_{d=1} = \mathbf{1} \wedge \partial_x \wedge \ldots \wedge \partial_x^{N-1}\bigr)$, here $N = k+1$, of the Lie algebra $\mathfrak{sl}(2)$, itself realised on the line $\mathbb{R}^1$ by using quadratic vector fields. Let us begin by recalling that classical construction.

Example 1 (realisation of $\mathfrak{sl}(2)$ by polynomial vector fields)**.** Let $e := 1$, $h := -2x$, and $f := -x^2$ be monomials on the real line $\mathbb{R} \ni x$. Take the Wronskian determinant $W^{0,1} = \mathbf{1} \wedge \partial/\partial x$ of two functions on $\mathbb{R}$ as the bi-linear antisymmetric bracket. Then we readily see that

$$W^{0,1}(h, e) = 2e, \qquad W^{0,1}(h, f) = -2f, \qquad \text{and} \qquad W^{0,1}(e, f) = h, \tag{1}$$

which are the standard relations for $\mathfrak{sl}(2)$ w.r.t the Chevalley–Cartan basis; a verification that the Wronskian determinant on $C^\infty(\mathbb{R})$ satisfies the Jacobi identity is straightforward.

We thus realise $\mathfrak{sl}(2)$ by using three vector fields with polynomial coefficients over $\Bbbk = \mathbb{C}$ (or $\Bbbk = \mathbb{R}$).

By increasing the differential order of the Wronskian $W^{0,1,\ldots,k} = \mathbf{1} \wedge \partial_x \wedge \cdots \wedge \partial_x^k$ to arbitrary $k \geqslant 1$, and now letting $N = k+1$, we endow the space of polynomials $\Bbbk[x]$ in one variable x with an N-linear (over the number field $\Bbbk$) antisymmetric bracket:

$$W^{0,1,\ldots,N-1}(p_1, \ldots, p_N) = \det\Bigl(\partial^{i-1} p_j / \partial x^{i-1}\Bigr)_{ij}(x) \tag{2}$$

for $p_1, \ldots, p_N \in \Bbbk[x]$ and $i, j \in \{1, \ldots, N\}$. From [2] we recall that the N-ary Wronskian brackets satisfy the N-ary Jacobi identities for strong homotopy SH-Lie algebras (see [12, 13], and [7] for definitions and details). In particular, we know a class of finite-dimensional N-ary SH-Lie algebras generalising $\mathfrak{sl}(2)$ with its binary structure from Example 1.

Example 2 (from [6])**.** Over dimension $d = 1$, let x be the coordinate on $\Bbbk = \mathbb{R}$. The Wronskian $W^{0,1,\ldots,N-1} = \mathbf{1} \wedge \partial_x \wedge \cdots \wedge \partial_x^k$ of differential order $k = N-1$ acting on the subspace of polynomials of degree at most N, so $\mathcal{A} = \Bbbk_N[x] \subsetneq \Bbbk[x]$, makes $\mathcal{A}$ into an $(N+1)$-dimensional SH-Lie algebra: for any degree $\ell \in \{0, 1, \ldots, N\}$ we have

$$W^{0,1,\ldots,N-1}_d\left(1, \ldots, \widehat{\frac{x^\ell}{\ell!}}, \ldots, \frac{x^N}{N!}\right) = \frac{x^{N-\ell}}{(N-\ell)!}, \tag{3}$$

which makes all the non-automatically-zero structure constants equal to ± 1 in the basis $\{x^m/m! : m \in \{0, 1, \ldots, N\}\}$.

Now, we increase the dimension to $d = 2$ and we let x, y be the Cartesian coordinates on $\mathbb{R}^2$. Consider the generalised Wronskian $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$ of differential order $k = 1$: for any smooth functions $f, g, h \in C^\infty(\mathbb{R}^2)$, put

$$W^{k=1}_{d=2}(f,g,h)(x,y) = \det\begin{pmatrix} f & g & h \\ f_x & g_x & h_x \\ f_y & g_y & h_y \end{pmatrix}(x,y). \tag{4}$$

As seen from [6] (also [8]), this ternary skew bracket equips the space of smooth functions $C^\infty(\mathbb{R}^2)$ with the SH-Lie algebra structure.

Until now we knew three examples of finite-dimensional polynomial SH-Lie algebras with the ternary bracket (4); here they are:

Example 3. Over dimension $d = 2$, take the generalised Wronskian $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$ of differential order $k = 1$ and the ternary bracket $[\cdot,\cdot,\cdot]$ on the space of smooth functions. Now in $C^\infty(\mathbb{R}^2) \supsetneq \mathbb{R}[x,y]$, consider three distinct four-dimensional subspaces of degree-two polynomials spanned by $1, x, y$ and a specific monomial of degree two:

- $\mathcal{A}(x^2) = \operatorname{span}_\mathbb{R}\langle 1, x, y, x^2\rangle$; the ternary commutation relations are
$$[1,x,y] = 1, \qquad [1,x,x^2] = 0, \qquad [1,y,x^2] = -2x, \qquad [x,y,x^2] = -x^2;$$
- $\mathcal{A}(xy) = \operatorname{span}_\mathbb{R}\langle 1, x, y, xy\rangle$; the ternary brackets are (cf. [6])
$$[1,x,y] = 1, \qquad [1,x,xy] = x, \qquad [1,y,xy] = -y, \qquad [x,y,xy] = -xy;$$
- $\mathcal{A}(y^2) = \operatorname{span}_\mathbb{R}\langle 1, x, y, y^2\rangle$; we have that
$$[1,x,y] = 1, \qquad [1,x,y^2] = 2y, \qquad [1,y,y^2] = 0, \qquad [x,y,y^2] = -y^2.$$

All three polynomial SH-Lie algebras are closed under the ternary bracket and thus four-dimensional.

Remark 1. The SH-Lie algebras $\Bbbk_N[x]$ and $\mathcal{A}(xy)$ from Examples 2 and 3, respectively, are *perfect*, that is, the N-ary bracket has surjective span:[1] $\operatorname{span}_\mathbb{R}[\mathcal{A},\mathcal{A},\mathcal{A}] = \mathcal{A}$. In contrast, the other algebras in Example 3 are not perfect (as the ternary bracket is missing a monomial from its image).

In Proposition 15 below it will be proven that the two algebras – over dimension $d = 1$, $\Bbbk_N[x]$ with $k = N - 1$, and $\mathcal{A}(xy)$ over $d = 2$ – are the only examples of perfect finite-dimensional polynomial SH-Lie algebras with complete generalised Wronskians as the N-ary brackets, *under the extra assumption* that the underlying vector space is spanned by *pure monomials*.

Finally, we have the fourth class of finite-dimensional examples.

Example 4. Fix the dimension $d \geqslant 1$ and differential order $k \geqslant 1$, and take the span,

$$\operatorname{span}_\Bbbk\{1, x^1, \ldots, x^d, (x^1)^2, x^1x^2, \ldots, (x^d)^k\}, \tag{5}$$

of all monomials up to – including but not exceeding – degree k. We recall that there are $N = \binom{d+k}{k}$ such monomials; this obviously is the number of arguments in the complete generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$, as the derivations in it stand in a bijective correspondence with the monomials in our set (cf. Definition 4 in Preliminaries).

[1] In literature the notation $[\mathfrak{g},\mathfrak{g}]$ is already understood to be the span, i.e. $[\mathfrak{g},\mathfrak{g}] := \operatorname{span}_\Bbbk[\mathfrak{g},\mathfrak{g}]$. An alternative notation for the span is $\langle[\mathfrak{g},\mathfrak{g}]\rangle$.

Up to a reshuffle of all pairwise distinct arguments, the only non-zero bracket then is

$$W^{k\geqslant 1}_{d\geqslant 1}\Big(1, x^1, ..., x^d, \frac{(x^1)^2}{2!}, \frac{x^1x^2}{1!1!}, ..., \frac{(x^d)^k}{k!}\Big) = +1. \tag{6}$$

This N-dimensional polynomial SH-Lie algebra is trivially closed, and it is not perfect for all differential orders $k \geqslant 1$ and dimensions $d \geqslant 1$.

In Theorem 7 we shall prove the following Lemma as a by-product.

Lemma 1. Over any base dimension $d \geqslant 1$ and any differential order $k \geqslant 1$, the complete generalised Wronskian determinant of monomials from $\Bbbk[\boldsymbol{x}]$ is again a monomial.
• By multilinearity then, the Wronskian determinant of polynomials is again a polynomial in $\Bbbk[\boldsymbol{x}]$.

Acting as the N-ary bracket, the complete generalised Wronskian determinant $W^{k\geqslant 1}_{d\geqslant 1} = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}...\boldsymbol{x}}$ endows the space $\Bbbk[\boldsymbol{x}]$ of polynomials over $\Bbbk^d$ with a structure of strong homotopy Lie algebra (see [6, 7] and references therein, also [12]).

Definition 1. Let $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}] = \Bbbk[x^1, ..., x^d]$ be a subalgebra of the N-ary SH-Lie algebra of polynomials $\Bbbk[\boldsymbol{x}]$ with the Wronskian determinant of differential order k over dimension d as N-ary bracket, $N = \binom{d+k}{k}$; i.e. a vector subspace closed under the Wronskian N-ary bracket $[\cdot, ..., \cdot] = W^k_d(\cdot, ..., \cdot)$ on $\Bbbk[\boldsymbol{x}]$. The *dimension* of $\mathcal{A}$ is the dimension of the subspace $\mathcal{A}$ viewed as a vector space over $\Bbbk$. In particular, $\dim_{\Bbbk}(\Bbbk[\boldsymbol{x}]) = +\infty$.

Definition 2. A Lie algebra $\mathfrak{g} = (V, [\cdot, \cdot])$ over $\Bbbk$ is called *perfect* if $\operatorname{span}_{\Bbbk}[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Now, in our case of N-ary structures, specifically, $N = \binom{d+k}{k}$, we say that the polynomial N-ary SH-Lie algebra $\mathcal{A}$ with the Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$ as the bracket is *perfect* if the bracket spans the whole algebra: $\operatorname{span}_{\Bbbk} W^k_d(\mathcal{A}, ..., \mathcal{A}) = \mathcal{A}$.

Any simple or semi-simple Lie algebra is immediately perfect. Finite-dimensional (semi-) simple complex Lie algebras are of great interest due to their classification by Dynkin diagrams and construction of their commutator tables from root systems. This leads to our main research problem.

Research problem. For every dimension $d \geqslant 1$ and every differential order $k \geqslant 1$ do there exist and, if so, what are all (perfect) *finite-dimensional* polynomial N-ary SH-Lie, $N = \binom{d+k}{k}$, subalgebras $\mathcal{A} \subseteq \Bbbk[x^1, ..., x^d]$ with the (complete) generalised Wronskian determinant $W^k_d = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}...\boldsymbol{x}}$ as the N-ary Lie bracket?

List of notation. • $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, base field;
• $d = \dim_{\mathbb{R}} \mathbb{R}^d = \dim_{\Bbbk} \Bbbk^d$;
• $\boldsymbol{x} = (x^1, ..., x^d)$ Cartesian coordinates on $\mathbb{R}^d$;
• $\Bbbk[x^1, ..., x^d]$, space of polynomials over $\mathbb{R}^d$;
• $\Bbbk_k[x^1, ..., x^d]$ polynomials of degree $\leqslant k$;
• $N = \binom{d+k}{d} = \binom{d+k}{k}$, arity – Definition 3;
• perfect SH-Lie algebra $\mathcal{A} = \operatorname{span}_{\Bbbk}[\mathcal{A}, ..., \mathcal{A}]$;
• power d-tuple $\vec{\boldsymbol{r}} = (r^1, ..., r^d) \in \mathbb{Q}^d$;
• $\boldsymbol{x}^{\vec{\boldsymbol{r}}} = (x^1)^{r^1} \cdots (x^d)^{r^d}$ arbitrary monomial;
• $\partial^{|\vec{\boldsymbol{r}}|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}} = \partial^{r^1}/\partial(x^1)^{r^1} \circ \cdots \circ \partial^{r^d}/\partial(x^d)^{r^d}$ differential operator – cf. Eq. (12);
• $\boldsymbol{x}^{\vec{\boldsymbol{r}}} \leftrightarrows \partial^{|\vec{\boldsymbol{r}}|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ – differential operators vs divided power monomials – Definition 4;
• operator in the jth row of the Wronskian: $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ – Definition 4;
• standard monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ – Definition 4;
• $\det \mathrm{Van}^{0,1,...,N-1}_{d=1}$ ordinary Vandermonde det.;
• $\det \mathrm{Van}^{k\geqslant 1}_{d\geqslant 1}$ generalised – Definition 5;
• $\det \Delta^d_k$, quasi-triangular form – Definition 6;
• $\boldsymbol{m}^{\vec{\boldsymbol{r}}} = (m_1)^{r^1} \cdots (m_d)^{r^d}$;
• $\vec{\boldsymbol{k}} = \boldsymbol{k} \in \mathbb{Q}^d$ power d-tuple;
• $kN/(d+1)$ degree shift – Lemma 6;

• deficiency condition – Proposition 9 (*iii*);
• $s = k/(d+1) \cdot N/(N-1)$ – Remark 8;
• consistent algebra, $\Bbbk_k[\boldsymbol{x}] \subseteq \mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$, trivial, lonely, chubby, tall, lanky algebras – Definition 8.

Preliminaries.

Let us recall some notions and fix notation.

Notation 1. Let $\Bbbk = \mathbb{R}$ (or $\mathbb{C}$) be the ground field; let $\boldsymbol{x} = (x^1, ..., x^d)$ be the (Cartesian) coordinates on $\mathbb{R}^d$, where d is the base dimension. By convention, put $\partial_{x^i} = \partial/\partial x^i$ and $\partial_{\boldsymbol{x}} = \partial_{x^1} \wedge \cdots \wedge \partial_{x^d}$; also, put $\partial^2_{\boldsymbol{xx}} = \partial^2_{x^1x^1} \wedge \cdots \wedge \partial^2_{x^ix^j} \wedge \cdots \wedge \partial^2_{x^dx^d}$ and so on until $\partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}} = \partial^k_{x^1\ldots x^1} \wedge \cdots \wedge \partial^k_{x^d\ldots x^d}$. Denote the identity operator by $\mathbf{1}$.

• Let $\vec{\boldsymbol{r}} = (r^1, ..., r^d)$ be a d-*tuple* (also called multi-index) in $\mathbb{N}^d_{\geqslant 0}$ (occasionally in $\mathbb{Z}^d$ or $\mathbb{Q}^d$). We denote by $\boldsymbol{x}^{\vec{\boldsymbol{r}}}$ the monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}} = (x^1)^{r^1} \cdots (x^d)^{r^d}$; and denote by $\partial^{|\vec{\boldsymbol{r}}|}/\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}}$ the differential operator

$$\frac{\partial^{|\vec{\boldsymbol{r}}|}}{\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}}} := \frac{\partial^{r^1}}{\partial(x^1)^{r^1}} \cdots \frac{\partial^{r^d}}{\partial(x^d)^{r^d}}. \tag{7}$$

For instance, the 2-tuple $\vec{\boldsymbol{r}} = (2,1)$ gives $\partial^3/\partial x^2\partial y$, where $x := x^1$ and $y := x^2$. By definition, put $\vec{\boldsymbol{r}}! = r^1! \cdots r^d!$. Wherever convenient, we shall lower the indices of the power d-tuple $\vec{\boldsymbol{r}}$ without change of meaning; for instance, setting $\vec{\boldsymbol{r}} = \vec{\boldsymbol{n}} = (n_1, ..., n_d)$, the monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}}$ becomes $(x^1)^{n_1} \cdots (x^d)^{n_d}$.

• Arbitrary monomials will usually be denoted by $a(\boldsymbol{x})$, whereas polynomials by $p(\boldsymbol{x})$ or $q(\boldsymbol{x})$; however, exceptions are possible (in §2.3; cf. Footnote 12 for the reasoning).

Definition 3. Consider the algebra of smooth functions functions over coordinates $\boldsymbol{x} = (x^1, ..., x^d)$; we say that $d \geqslant 1$ is the *dimension*. Let the *total differential order* be $k \geqslant 1$, and set $N = \binom{d+k}{k} = \binom{d+k}{d}$. The N-linear (over $\Bbbk$) totally antisymmetric, $C^\infty(\mathbb{R}^d)$-valued differential operator $W^k_d = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$ is the *complete generalised Wronskian determinant* of differential order k over dimension d.

Notation 2. Denote by $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ the differential operator in the jth row, $j \in \{1, ..., N\}$, of the Wronskian W^k_d. In this notation, $W^k_d = \bigwedge_{j=1}^N \partial^{|\vec{\boldsymbol{r}}_j|}/\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$. We say that r^i_j is the *differential order* in coordinate x^i and $r_j = |\vec{\boldsymbol{r}}_j| = r^1_j + \cdots + r^d_j$ is the *total differential order* of the differential operator $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$.

Example 5. Take smooth functions $f, g, h \in C^\infty(\mathbb{R}^1)$ and put $W^{k=1}_{d=1} = \mathbf{1} \wedge \partial/\partial x$ and $W^{k=2}_{d=1} = \mathbf{1} \wedge \partial/\partial x \wedge \partial^2/\partial x^2$. Then the Wronskians,

$$W^{k=1}_{d=1}(f,g)(x) = \det\begin{vmatrix} f & g \\ f' & g' \end{vmatrix}(x), \qquad W^{k=2}_{d=1}(f,g,h)(x) = \det\begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}(x), \tag{8}$$

produce smooth functions on the real line $\mathbb{R}^1$.

Alternatively, let the dimension be $d = 2$ and the differential order be $k = 1$. Then for smooth functions $f, g, h \in C^\infty(\mathbb{R}^2)$ we have

$$W^{k=1}_{d=2}(f,g,h)(\boldsymbol{x}) = \det\begin{vmatrix} f & g & h \\ f_x & g_x & h_x \\ f_y & g_y & h_y \end{vmatrix}(\boldsymbol{x}), \tag{9}$$

where $f_x := \partial f/\partial x$ for $x = x^1$ and $y = x^2$ and which is again a smooth function over $\mathbb{R}^2$.

In Notation 1 the above Wronskians become

$$W^{k=1}_{d=1} = \mathbf{1} \wedge \partial_x = \frac{\partial^{|\vec{\boldsymbol{r}}_1|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_1}} \wedge \frac{\partial^{|\vec{\boldsymbol{r}}_2|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}, \qquad W^{k=1}_{d=2} = \frac{\partial^{|\vec{\boldsymbol{r}}_1|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_1}} \wedge \frac{\partial^{|\vec{\boldsymbol{r}}_2|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_2}} \wedge \frac{\partial^{|\vec{\boldsymbol{r}}_3|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_3}}, \tag{10}$$

where we observe that in all cases $\partial^{|\vec{\boldsymbol{r}}_1|}\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_1}$ is the identity $\mathbf{1}$; by definition, then $r_1 = |\vec{\boldsymbol{r}}_1| = 0$ and $r_1^i = 0$ for all $i \in \{1, ..., d\}$. The second operator is always of order 1, namely, $r_2 = |\vec{\boldsymbol{r}}_2| = 1$ with $r_2^i = 1$ for a single $i \in \{1, ..., d\}$ and all others vanishing: $r_2^{i'} = 0$ for $i' \neq i$.

Let us recall a natural correspondence between monic single-term differential operators, like $\partial^3/\partial x^2 \partial y$, and monomials (resp., $x^2 y$) over $\mathbb{R}^d$, and functions $(m^1, ..., m^d) \mapsto (m^1)^2 \cdot m^2$ from $\mathbb{Q}^d$.

Definition 4 (divided power monomial and standard monomial)**.** In a *fixed* system of coordinates $\boldsymbol{x} = (x^1, ..., x^d)$ on $\mathbb{R}^d$, let $D = \partial^{|\vec{\boldsymbol{r}}|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}}$ be a single-term monic differential operator of total differential order $r = |\vec{\boldsymbol{r}}|$. The *divided power monomial* $a(\boldsymbol{x})$ corresponding to D is, by definition, the (unique) monomial in $\Bbbk[\boldsymbol{x}]$ satisfying $D(a)(\boldsymbol{x}) \equiv 1$, the unit constant polynomial.

- The formula $a(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{r}}}/\vec{\boldsymbol{r}}! = (x^1)^{r^1} \cdots (x^d)^{r^d}/r^1! \cdots r^d!$ obviously satisfies $D(a)(\boldsymbol{x}) \equiv 1$ for $D = \partial^{|\vec{\boldsymbol{r}}|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}}$ as above with any power d-tuple (multi index) $\vec{\boldsymbol{r}}$ in $\mathbb{N}^d_{\geqslant 0}$. In particular, for $D = \partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$, the differential operator of the jth row of the Wronskian $W^k_d = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$ (of differential order $k \geqslant 1$ over dimension $d \geqslant 1$), the divided power monomial corresponding to $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ is $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!$.

The *standard set* of monomials w.r.t. the Wronskian W^k_d is $\{\boldsymbol{x}^{\vec{\boldsymbol{r}}_1}/\vec{\boldsymbol{r}}^1, ..., \boldsymbol{x}^{\vec{\boldsymbol{r}}_N}/\vec{\boldsymbol{r}}_N!\}$, and the divided power monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!$ is called the *standard monomial corresponding to the jth row of the Wronskian*.

- The correspondence also sends a *monomial* to its evaluation map on $\boldsymbol{m} \in \mathbb{Q}^d$:

$$\begin{aligned} \Bbbk[\boldsymbol{x}] &\to (\mathbb{Q}^d \to \mathbb{Q}) \\ \boldsymbol{x}^{\vec{\boldsymbol{r}}} &\mapsto (\boldsymbol{m} \mapsto \boldsymbol{m}^{\vec{\boldsymbol{r}}}). \end{aligned} \tag{11}$$

Basically, $x^2 y^3$ becomes the map $\mathbb{Q}^{d=2} \to \mathbb{Q}$ given by $(m_1, m_2) \mapsto (m_1)^2 (m_2)^3$.

Let $a(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{n}}} = (x^1)^{n_1} \cdots (x^d)^{n_d}$ be a monomial with power d-tuple $\vec{\boldsymbol{n}} \in \mathbb{N}^d_{\geqslant 0}$. The partial derivative of $a(\boldsymbol{x})$ with respect to x^1 is then $\partial a/\partial x^i = n_i \cdot a/x^i$; this extends to higher-degree differential operators $\partial^{|\vec{\boldsymbol{r}}|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}}$ by

$$\frac{\partial^{|\vec{\boldsymbol{r}}|} a}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}}}(\boldsymbol{x}) = \frac{a}{\boldsymbol{x}^{\vec{\boldsymbol{r}}}} \cdot \prod_{i=1}^{d} \left\{ \prod_{\ell=0}^{r^i - 1} (n_i - \ell) \right\}. \tag{12}$$

If, for some $i \in \{1, ..., d\}$ the differential order r^i exceeds the degree of x^i in $a(\boldsymbol{x})$, i.e. $r^i > n_i$, then the product $\prod_{\ell=0}^{r^i-1}(n_i - \ell)$ vanishes, and therefore the 'negative' degree of $a/(x^i)^{r_i}$ in x^i is fictitious as the entire r.-h.s. of (12) is identically zero.

1. Multivariate Vandermonde determinants are coefficients of multivariate Wronskians

In this section we generalise the Vandermonde determinant to an arbitrary base dimension $d \geqslant 1$ in a way that respects the structure of the complete generalised Wronskian determinant $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$ of differential order $k \geqslant 1$ over dimension $d \geqslant 1$. The complete generalised Vandermonde determinant of degree k over dimension d appears (see Theorem 7 on p. 30) as the coefficient of the monomial resulting from a Wronskian determinant of monomial arguments; the arguments of this Vandermonde determinant are the power d-tuples of the monomial arguments in the Wronskian. We use the correspondence between monic single-term differential operators, monomials, and evaluation functions on $\mathbb{Q}^d$: for example,

$$\partial^5/\partial x^2\partial y^3 \leftrightarrows x^2y^3 \leftrightarrows \{(m_1,\ldots,m_d) \mapsto (m_1)^2(m_2)^3\}, \tag{13}$$

where $\boldsymbol{m} = (m^1,\ldots,m^d)$ is an argument of the Vandermonde determinant; see Definition 4.

The main theorem in this section is a generalisation of Theorem 13 from [6] (see Theorem 7) specifically from over dimension 1 to arbitrary dimension $d \geqslant 1$; that is, the (complete) generalised Wronskian determinant (of differential order $k \geqslant 1$ over dimension $d \geqslant 1$) of monomial arguments is again a monomial with the coefficient given by the complete generalised Vandermonde determinant of the power d-tuples of the monomial arguments. As part of the theorem, we give an alternative form of the generalised Vandermonde determinant, which we call quasi-triangular.

Finally, we shall give three sufficient conditions of the complete generalised Vandermonde determinant (hence also the Wronskian) to vanish (see Proposition 9 on p. 32); of these, one is non-trivial.

Recall the definition of the (ordinary) Vandermonde determinant (over dimension $d = 1$) and an old result [6] about the Wronskians of monomials.

Theorem–Definition 2 (Vandermonde determinant)**.** Take numbers $m_1,\ldots,m_N$ from $\mathbb{N}_{\geqslant 0}$ (or $\mathbb{Z}$, or $\mathbb{Q}$); their Vandermonde determinant is

$$\det \mathrm{Van}_{d=1}^{0,1,\ldots,N-1}(m_1,\ldots,m_N) = \det \begin{vmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_N \\ \vdots & & \vdots \\ m_1^{N-1} & \cdots & m_N^{N-1} \end{vmatrix} = \prod_{1\leqslant i<j\leqslant N} (m_j - m_i), \tag{14}$$

i.e. the determinant fully factorises to a product of linear factors in $m_1,\ldots,m_N$.

Theorem 3 (Theorem 13 from [6])**.** *Let $m_1,\ldots,m_N \in \mathbb{N}_{\geqslant 0}$ (or $\mathbb{Z}$, or $\mathbb{Q}$) be constants and set $m = m_1 + \cdots + m_N$; then the Wronskian determinant $W_{d=1}^{0,1,\ldots,N-1} := W_{d=1}^{k=N-1}$ of (generalised) monomial arguments becomes the (generalised) monomial*

$$W_{d=1}^{0,1,\ldots,N-1}(x^{m_1},\ldots,x^{m_N}) = \prod_{1\leqslant i<j\leqslant N} (m_j - m_i)x^{m-N(N-1)/2}, \tag{15}$$

where the coefficient is the <u>Vandermonde</u> *determinant of their powers.*

Remark 2 (generalisation of the Witt algebra, [7]). The (complex, infinite-dimensional) Witt algebra is given by differential operators with Laurent polynomial coefficients ordered by degree, $a_i = x^{i+1}$, and with the Lie bracket satisfying the commutation relations $[a_i, a_j] =$

$(j-i)a_{i+j}$ for all $i,j\in\mathbb{Z}$. Its (strong) homotopy L_∞ deformations by Wronskians increase the bracket arity; with the proper index shift $a_i = x^{i+N/2}$, by Theorem 2, the N-ary bracket $[\cdot,\ldots,\cdot]_N$ given by the (ordinary) Wronskian determinant $W^{0,1,\ldots,N-1}_{d=1}$, satisfies the commutation relations

$$[a_{i_1},\ldots,a_{i_N}]_N = \det \mathrm{Van}^{0,1,\ldots,N-1}_{d=1}(i_1,\ldots,i_N)\cdot a_{i_1+\cdots+i_N}, \tag{16}$$

where the structure constants, the Vandermonde determinants, are totally antisymmetric w.r.t their arguments.

Before we give a natural multivariate generalisation of the Vandermonde determinants w.r.t the Wronskian determinant, let us have some motivating examples, which will illustrate all steps in the proof of Theorem 7 below.

Example 6 ($k=1, d=2$)**.** Let the dimension be $d=2$ and the differential order be $k=1$; set $N=\binom{d+k}{k}=3$. Consider $W^{d=2}_{k=1}=\mathbf{1}\wedge\partial/\partial x\wedge\partial/\partial y$ as the ternary ($N=3$) bracket $[\cdot,\cdot,\cdot]$. Let $\boldsymbol{x}^{\boldsymbol{m}_1}$, $\boldsymbol{x}^{\boldsymbol{m}_2}$, and $\boldsymbol{x}^{\boldsymbol{m}_3}$ be arbitrary monomials ($m^i_j\in\mathbb{N}_{\geqslant 0}$ for all i,j) or generalised monomials ($m^i_j\in\mathbb{Z}$ or $\mathbb{Q}$). For the ease of notation let $\vec{\boldsymbol{m}}_1=(k_1,k_2)$, $\vec{\boldsymbol{m}}_2=(\ell_1,\ell_2)$, and $\vec{\boldsymbol{m}}_3=(n_1,n_2)$. Then their bracket becomes

$$[\boldsymbol{x}^{\boldsymbol{m}_1},\boldsymbol{x}^{\boldsymbol{m}_2},\boldsymbol{x}^{\boldsymbol{m}_3}] = \det\begin{vmatrix} x^{k_1}y^{k_2} & x^{\ell_1}y^{\ell_2} & x^{n_1}y^{n_2}\\ k_1x^{k_1-1}y^{k_2} & \ell_1x^{\ell_1-1}y^{\ell_2} & n_1x^{n_1-1}y^{n_2}\\ k_2x^{k_1}y^{k_2-1} & \ell_2x^{\ell_1}y^{\ell_2-1} & n_2x^{n_1}y^{n_2-1}\end{vmatrix}$$

$$= x^{k_1-1}y^{k_2-1}\cdot x^{\ell_1-1}y^{\ell_2-1}\cdot x^{n_1-1}y^{n_2-1}\cdot\det\begin{vmatrix} xy & xy & xy\\ k_1y & \ell_1y & n_1y\\ k_2x & \ell_2x & n_2x\end{vmatrix},$$

by gathering common powers per column,

$$= x^{k_1+\ell_1+m_1-3}y^{k_2+\ell_2+n_2-3}\cdot xy\cdot y\cdot x\cdot\det\begin{vmatrix}1&1&1\\k_1&\ell_1&m_1\\k_2&\ell_2&n_2\end{vmatrix},$$

by gathering common monomials per row, leaving just the determinant of the number matrix

$$= x^{k_1+\ell_1+n_1-1}y^{k_2+\ell_2+n_2-1}\cdot\det\begin{vmatrix}1&1&1\\k_1&\ell_1&n_1\\k_2&\ell_2&n_2\end{vmatrix}$$

$$= \boldsymbol{x}^{\vec{\boldsymbol{m}}_1+\vec{\boldsymbol{m}}_2+\vec{\boldsymbol{m}}_3-\mathbf{1}}\cdot\det\begin{vmatrix}1&1&1\\k_1&\ell_1&n_1\\k_2&\ell_2&n_2\end{vmatrix},$$

where $\mathbf{1}=(1,1)$ is the all-ones vector. We see that the result consists of a monomial in x (resp. y) with their powers given by the sum of arguments' degrees in x (resp. y) minus a shift, and the coefficient is a *generalisation* of the Vandermonde determinant: instead of degrees' (of x) squares $(k_1)^2,(\ell_1)^2,(n_1)^2$ in the 3rd row, it contains the powers of y, namely, k_2,ℓ_2,n_2.

Notation 3. In the above Example 6, we have the coordinates $x^1=x$ and $x^2=y$, of which we take powers to form monomials like $(x^1)^{m^1_j}\cdot(x^2)^{m^2_j}$. By convention (see Notation 1 on

p. 23), $(x^1)^{m^1} \cdot (x^2)^{m^2} = \boldsymbol{x}^{\vec{\boldsymbol{m}}}$ over the base dimension d (here $d = 2$) and $\vec{\boldsymbol{m}} \in \mathbb{N}^d_{\geqslant 0}$ (or $\mathbb{Z}^d$, or $\mathbb{Q}^d$).

For the (complete) generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$ of arity $N = \binom{d+k}{k}$ we now take the monomials $(x^1)^{m^1_j} \cdots (x^d)^{m^d_j} = \boldsymbol{x}^{\vec{\boldsymbol{m}}_j}$ with $j \in \{1, ..., N\}$ as its arguments. For every j and its respective power d-tuple $\vec{\boldsymbol{m}}_j = (m^1_j, ..., m^d_j)$, we put by definition $\boldsymbol{m}_j = \vec{\boldsymbol{m}}_j$.

Next, consider the nth row in the Wronskian: in it, each argument is differentiated by the differential operator $\partial^{|\vec{\boldsymbol{r}}_n|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_n}$ (see Notation 2 on p. 23); by Definition 3 on p. 23, to this operator there corresponds a monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}_n}$, which we take as the (evaluation) function $\mathbb{N}^d_{\geqslant 0} \to \mathbb{N}_{\geqslant 0}$ (or with $\mathbb{N}_{\geqslant 0}$ replaced by $\mathbb{Z}$ or $\mathbb{Q}$) given by the formula $(m^1, ..., m^d) \mapsto \boldsymbol{m}^{\vec{\boldsymbol{r}}_j} = (m^1)^{r^1_j} \cdots (m^d)^{r^d_j}$. For instance, $\partial^3/\partial x^2 \partial y \leftrightarrows x^2 y \leftrightarrows \{(m^1, m^2) \mapsto (m^1)^2\, m^2\}$. In our notation, $\boldsymbol{m}_j^{\vec{\boldsymbol{r}}_n}$ means raising the powers of base variables m^α_j to the respective degrees r^α_n by which the arguments in the Wronskian are differentiated in the nth row. Here $\vec{\boldsymbol{r}}_1 = (0, ..., 0)$ gives $\boldsymbol{m}_j^{\vec{\boldsymbol{r}}_1} = 1$, the unit.

Definition 5 (generalised Vandermonde determinant)**.** Let the dimension be $d \geqslant 1$ and the total degree be $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let $\partial^{|\vec{\boldsymbol{r}}_n|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ be the differential operator from the nth row in the Wronskian $W^k_d = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$; here $\vec{\boldsymbol{r}}_n$ is its corresponding power d-tuple (multi-index). The *complete generalised Vandermonde determinant* of degree k over dimension d w.r.t the complete generalised Wronskian W^k_d of power d-tuples $\boldsymbol{m}_1, ..., \boldsymbol{m}_N \in \mathbb{Q}^d$ is the determinant

$$\det \mathrm{Van}^k_d(\boldsymbol{m}_1, ..., \boldsymbol{m}_N) = \det \begin{pmatrix} \boldsymbol{m}_1^{\vec{\boldsymbol{r}}_1} & \cdots & \boldsymbol{m}_N^{\vec{\boldsymbol{r}}_1} \\ \vdots & & \vdots \\ \boldsymbol{m}_1^{\vec{\boldsymbol{r}}_N} & \cdots & \boldsymbol{m}_N^{\vec{\boldsymbol{r}}_N} \end{pmatrix}, \tag{17}$$

where in our notation $\boldsymbol{m}_j^{\vec{\boldsymbol{r}}_n} = (m^1_j)^{r^1_n} \cdots (m^d_j)^{r^d_n}$ and the first row consists of only units: $\boldsymbol{m}_j^{\vec{\boldsymbol{r}}_1} = 1$. Let us remember that the formula of the complete generalised Vandermonde determinant is relative to the complete generalised Wronskian determinant.

Example 7. With the notation as above,

$$\det \mathrm{Van}^{k=N-1}_{d=1}(m_1, ..., m_N) = \det \begin{pmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_N \\ (m_1)^2 & \cdots & (m_N)^2 \\ \vdots & & \vdots \\ (m_1)^{N-1} & \cdots & (m_N)^{N-1} \end{pmatrix}, \tag{18}$$

namely, in dimension $d = 1$ we recover the ordinary Vandermonde determinant. Now for higher dimensions:

$$\det \mathrm{Van}^{k=1}_{d=2}(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = \det \begin{pmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} = (b_1c_2 - b_2c_1) - (a_1c_2 - a_2c_2) + (a_1b_2 - a_2b_1) \tag{19}$$

as before in Example 6 with $\boldsymbol{a} = \boldsymbol{m}_1$, etc.; finally,

$$\det \mathrm{Van}_{d=2}^{k=2}(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d},\boldsymbol{e},\boldsymbol{f}) = \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_1 & b_1 & c_1 & d_1 & e_1 & f_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 & f_2 \\ a_1^2 & b_1^2 & c_1^2 & d_1^2 & e_1^2 & f_1^2 \\ a_1a_2 & b_1b_2 & c_1c_2 & d_1d_2 & e_1e_2 & f_1f_2 \\ a_2^2 & b_2^2 & c_2^2 & d_2^2 & e_2^2 & f_2^2, \end{pmatrix} \tag{20}$$

which is the determinant of a 6×6 matrix as $N = \binom{2+2}{2} = 6$. As a side remark, we note that (see [1, Example 2.8]) the determinant in (20) vanishes if and only if the six points in the Euclidean plane $\boldsymbol{a} = (a_1, a_2), \boldsymbol{b} = (b_1, b_2), ..., \boldsymbol{f} = (f_1, f_2) \in \mathbb{R}^2$ lie on a conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F$.

Remark 3. Unlike the full factorisation of the ordinary Vandermonde determinant, (14) on p. 25, the generalised Vandermonde determinant in (17) over dimension $d \geqslant 2$ does not immediately factorise into a product of linear terms. Yet for special choices of arguments we approach the full factorisation of the generalised Vandermonde determinant in §2 (see Theorem 13 on p. 37 and Theorem 16 on p. 44, where we calculate the structure constants for particular classes of polynomial algebras with complete generalised Wronskian determinants as N-ary brackets). See Problem 4 and the accompanying Comment on p. 49 for further comment.

Proposition 4 (translation invariance)**.** For any (power) d-tuples $\boldsymbol{m}_1, ..., \boldsymbol{m}_N \in \mathbb{Q}^d$ and any (homogeneous shift) d-tuple $\boldsymbol{s} \in \mathbb{Q}$ we have

$$\det \mathrm{Van}_d^k(\boldsymbol{m}_1 + \boldsymbol{s}, ..., \boldsymbol{m}_N + \boldsymbol{s}) = \det \mathrm{Van}_d^k(\boldsymbol{m}_1, ..., \boldsymbol{m}_N), \tag{21}$$

i.e. the complete generalised Vandermonde determinant over base dimension $d \geqslant 1$ and degree $k \geqslant 1$ remains, even when $d > 1$, translation invariant in the course of an homogeneous translation of all its arguments.

Proof sketch. The proof works by subtracting previous lines of the determinant, reappearing later with a common factor depending only on the shift $\boldsymbol{s}$. Indeed, expand any $(\boldsymbol{m}_n + \boldsymbol{s})^{\vec{\boldsymbol{r}}_j} = \boldsymbol{m}_n^{\vec{\boldsymbol{r}}_j} + (\cdots)$. For differential order 1 we cancel the garbage $(\cdots)$ by adding an s^i-multiple of the first row $(1, ..., 1)$. We cancel the garbage in all subsequent rows by adding s^i-multiplies of the previous rows. □

We work over $\Bbbk[x^1, ..., x^d]$ with N-ary bracket, $N = \binom{d+k}{d}$, given by the complete generalised Wronskian determinant W_d^k of differential order k over dimension d. Note that the rows of any differential order $0 \leqslant r \leqslant k$ form the complete set of monomials of degree r. For instance, over dimension $d = 2$ and differential order $k = 2$ in the complete Wronskian $W_{d=2}^{k=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y \wedge \partial_{xx}^2 \wedge \partial_{xy}^2 \wedge \partial_{yy}^2$, let us take degree $r = 2$: we obtain the divided power monomials $x^2/2!$, $xy/1!1!$, and $y^2/2!$. Note that these are all the basic monomials of degree 2 over dimension 2; the same is true for each degree r from the range $r \in \{0, 1, ..., k\}$. Hence the following lemma.

Lemma 5. In the complete generalised Wronskian $W_{d\geqslant 2}^{k\geqslant 1} = \bigwedge_{j=1}^N \partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ over base dimension $d \geqslant 2$ and differential order $k \geqslant 1$, the sum of differential operators' orders in a coordinate x^i, $i \in \{1, ..., d\}$, *does not depend on* i (and equals $\sum_{j=1}^N r_j^i$ by construction).

By the differential operator and monomial correspondence (basically, $\partial^3/\partial x^2 \partial y \leftrightarrows x^2 y$), the sum of degrees r^i_j is the same in the Vandermonde determinant relative to the Wronskian as above.

We now calculate this sum of degrees $\sum_{j=1}^N r^i_j$ in a coordinate x^i; it is the degrees (but not coordinates) that show up in the Vandermonde determinant. By the symmetry $\sum_{j=1}^N r^i_j = \sum_{j=1}^N r^{i'}_j$ of the *complete* generalised Wronskian W^k_d w.r.t base coordinates in it (here $i, i' \in \{1, ..., d\}$), the sum equals $\sum_{j=1}^N r^i_j = (1/d) \sum_{j=1}^N r_j$, where r_j is the total differential order of the operator in the jth row in the Wronskian W^k_d (see Definition 3 and Notation 2 on p. 23). By indexing the blocks of rows using their common total differential order r (e.g., $r = 2$ in dimension $d = 2$ gives three differentials $\partial^2_{xx}, \partial^2_{xy}$, and ∂^2_{yy}), we find that there are $\binom{d+r-1}{r}$ monomials of degree r over dimension d, hence equally many corresponding differential operators. Let us simplify the sum of binomial expressions within r_j.

Lemma 6 (combinatorial). For any integers $k, d \geq 1$ we have

$$\sum_{j=1}^N r_j = \sum_{r=0}^k r \binom{d+r-1}{r} = d \binom{d+k}{k-1} = \frac{kd}{d+1} N \in \mathbb{N}, \tag{22}$$

where $N = \binom{d+k}{d} = \binom{d+k}{k}$. Therefore for any $i \in \{1, ..., d\}$ we have

$$\sum_{j=1}^N r^i_j = \frac{1}{d} \sum_{j=1}^N r_j = \frac{kN}{d+1} \in \mathbb{N}. \tag{23}$$

(The *degree shift* in formula (23) and the *total degree shift* in formula (22) will reappear in Eq. (24) and many times onwards.)

Proof. First we rewrite, for the common degree $r \geq 1$:

$$r\binom{d+r-1}{r} = r\frac{(d+r-1)!}{r!\,(d-1)!} = \frac{(d+r-1)!}{(r-1)!\,(d-1)!}\,\frac{d}{d} = d\frac{(d+r-1)!}{(r-1)!\,d!} = d\binom{d+r-1}{d}.$$

Use the last equality to take d out of the sum,

$$\sum_{r=0}^k r\binom{d+r-1}{r} = 0 + \sum_{r=1}^k r\binom{d+r-1}{r} = d\sum_{r=1}^k \binom{d+r-1}{d} = d\binom{d+k}{d+1},$$

by the 'hockey-stick lemma' in the last equality. The result simplifies to $d\binom{d+k}{k-1}$, which we rewrite by the inductive property of combinations $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ as follows, setting $n := d + k + 1$,

$$\binom{d+k}{k-1} = \binom{d+k+1}{k} - \binom{d+k}{k} = \frac{d+k+1}{d+1}\frac{(d+k)!}{k!\,d!} - N = \frac{k}{d+1}N.$$

Combining all results above yields $\sum_{r=0}^k r\binom{d+r-1}{r} = \frac{kd}{d+1}N$ as desired. □

We are now ready to generalise Theorem 3 about the degree shift in the Wronskian determinant of monomials, now over dimension $d > 1$, and whereby the Vandermonde determinant of the arguments' powers (d-tuples) shows up as the coefficient; see formula (15) on p. 25. Let us set up the notation. Let $\vec{\boldsymbol{k}}_1, ..., \vec{\boldsymbol{k}}_N \in \mathbb{Q}^d$ be the power d-tuples; we put

$\boldsymbol{x}^{\vec{\boldsymbol{k}}_j} = (x^1)^{k_j^1} \cdots (x^d)^{k_j^d}$ as in Notation 1. We thus obtain N arbitrary (generalised) monomials over base dimension d; whenever all $k_j^i \in \mathbb{N}_{\geqslant 0}$, they are genuine monomials from $\Bbbk[x^1, ..., x^d]$. For a given number $s \in \mathbb{Q}$ (or in $\mathbb{N}_{\geqslant 0}$), the shift of a power multi-index $\vec{\boldsymbol{k}}$ occurs component-wise: let $\mathbf{1} = (1, ..., 1)$ so that $\vec{\boldsymbol{k}} + s\mathbf{1} := (k^1 + s, ..., k^d + s)$. Recall that the complete generalised Wronskian determinant is given, equivalently, by the two formulas: $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}} = \bigwedge_{j=1}^N \partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$.

Theorem 7 (generalised Vandermonde)**.** *Fix the base dimension $d \geqslant 1$ and degree $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let $\boldsymbol{x}^{\vec{\boldsymbol{k}}_1}, ..., \boldsymbol{x}^{\vec{\boldsymbol{k}}_N}$ be N arbitrary (generalised) monomials (their powers can be rational numbers). Set $\vec{\boldsymbol{k}} := \sum_{j=1}^N \vec{\boldsymbol{k}}_j$ with the component-wise summation: $k^i = \sum_{j=1}^N k_j^i$. Then the complete generalised Wronskian determinant of these monomials in d coordinates again is a monomial,*

$$W_d^k(\boldsymbol{x}^{\vec{\boldsymbol{k}}_1}, ..., \boldsymbol{x}^{\vec{\boldsymbol{k}}_N}) = \det \mathrm{Van}_d^k(\boldsymbol{k}_1, ..., \boldsymbol{k}_N) \cdot \boldsymbol{x}^{\vec{\boldsymbol{k}} - kN/(d+1)\mathbf{1}}, \tag{24}$$

and

$$\det \mathrm{Van}_d^k(\boldsymbol{k}_1, ..., \boldsymbol{k}_N) = \det \Delta_d^k(\boldsymbol{k}_1, ..., \boldsymbol{k}_N), \tag{25}$$

where the matrix $\Delta_d^k(\boldsymbol{k}_1, ..., \boldsymbol{k}_N)$ is given by

$$\Delta_d^k(\boldsymbol{k}_1, ..., \boldsymbol{k}_N)^{\text{row } j}_{\text{col } n} = \prod_{i=1}^d \left\{ \prod_{\ell=0}^{r_j^i - 1} (k_n^i - \ell) \right\}. \tag{26}$$

Definition 6. The matrix Δ_d^k is the *quasi-triangular* form of the Vandermonde matrix.

Note 4. The choice of the term, ‘quasi-triangular’, is motivated by the fact that whenever $\boldsymbol{k}_j = \boldsymbol{r}_j$ for all the lines of the generalised Wronskian determinant (so $j \in \{1, ..., N\}$), the matrix Δ_d^k is triangular. Let us emphasise that some other choice of the N monomials as arguments for W_d^k *may result in a non-triangular matrix* Δ_d^k; still, its determinant is equal to the Vandermonde determinant of the power d-tuples $\boldsymbol{k}_j$ of the monomial arguments.

Remark 5. Suppose all $\boldsymbol{x}^{\vec{\boldsymbol{k}}_j} \in \Bbbk[x]$ are actual monomials (i.e. $k_j^i > 0$ for all i, j). If, after the degree shift, for some coordinate x^i we have that the resulting degree is negative, $\sum_{j=1}^N k_j^i - kN/(d+1) < 0$, then the Vandermonde determinant must vanish identically. (Indeed, any derivatives and products of monomials are monomials; it is impossible to produce a Laurent monomial.) See also Eq. (12) on p. 24 and the surrounding discussion.

In Proposition 9 on p. 32 we list other cases when the Vandermonde determinant in front of the monomial equals zero.

Remark 6. Theorem 7 reduces to the old Theorem 3 on p. 25 when $d = 1$. Indeed, for $d = 1$ we have $N = k + 1$, hence $kN/(d + 1) = (N - 1) \cdot N/2 = N(N - 1)/2$.

Proof (of Theorem 7). The proof follows the same approach as in Example 6. For any monomial $a(\boldsymbol{x}) = c\boldsymbol{x}^{\vec{\boldsymbol{r}}}$, c is the coefficient and $\boldsymbol{x}^{\vec{\boldsymbol{r}}}$ the term; we solve for both in Eq. (24).

The differential operator $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ in the jth row of the Wronskian acting on the monomial $\boldsymbol{x}^{\vec{\boldsymbol{k}}_n}$ gives the term $\boldsymbol{x}^{\vec{\boldsymbol{k}}_n - \vec{\boldsymbol{r}}_j}$ and the coefficient in (26). As k is the highest total differential order of the Wronskian, $|\vec{\boldsymbol{r}}_j| \leqslant k$, factor $\boldsymbol{x}^{\vec{\boldsymbol{k}}_n - k\mathbf{1}}$ from the nth colmn of the determinant, which leaves $\boldsymbol{x}^{k\mathbf{1}-\vec{\boldsymbol{r}}_j}$, the same term in the jth row; factor it, leaving only numbers in the entries of the determinant. The overall term is thus $\boldsymbol{x}^{\vec{\boldsymbol{n}}}$, where the power d-tuple $\vec{\boldsymbol{n}}$ is given by

$$\vec{\boldsymbol{n}} = \sum_{n=1}^{N}(\vec{\boldsymbol{k}}_n - k\mathbf{1}) + \sum_{j=1}^{N}(k\mathbf{1} - \vec{\boldsymbol{r}}_j) = \sum_{j=1}^{N}\vec{\boldsymbol{k}}_j - \frac{kN}{d+1}\mathbf{1} = \vec{\boldsymbol{k}} - \frac{kN}{d+1}\mathbf{1}, \tag{27}$$

proving the statement in formula (24).

Expand each row in (26); the term of the highest degree in row j and column n is then $\boldsymbol{k}_n^{\vec{\boldsymbol{r}}_j}$, which is exactly the element in the Vandermonde determinant. As all lower-degree terms share an homogeneous coefficient in all rows, subtracting the preceding rows leaves only the top-degree term, proving that the two determinants are equal – (25). The proof is complete. □

Example 8. It is often much easier to compute the complete generalised Vandermonde determinant by using its quasi-triangular form. For instance, over dimension $d = 2$ with degree $k = 2$, and all standard monomials up to degree $k = 2$ as arguments (namely, $1, x, y, x^2, xy, y^2$ corresponding to the power d-tuples $\vec{\boldsymbol{r}}_1 = (0,0)$, followed by $(1,0)$, ..., $(1,1)$ until $\vec{\boldsymbol{r}}_6 = (0,2)$, respectively), we have

$$\det \mathrm{Van}_{d=2}^{k=2}(\vec{\boldsymbol{r}}_1,\ldots,\vec{\boldsymbol{r}}_6) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 2 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 2\\ 0 & 1 & 0 & 4 & 1 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 1 & 4 \end{pmatrix} = 4, \tag{28}$$

whereas the determinant of the quasi-triangular matrix $\Delta_d^k(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_6)$ is

$$\det \Delta_{d=2}^{k=2}(\vec{\boldsymbol{r}}_1,\ldots,\vec{\boldsymbol{r}}_6) = \det\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ & 1 & 0 & 2 & 1 & 0\\ & & 1 & 0 & 1 & 2\\ & & & 2 & 0 & 0\\ & \mathbf{0} & & & 1 & 0\\ & & & & & 2 \end{pmatrix} = 4. \tag{29}$$

Indeed, in this example the determinant equals the product of entries on the main diagonal.

By examining the mechanism by which the complete generalised Wronskian shifts the degree of each coördinate (see Theorem 7), we give the following definitions.

Definition 7 (shifts and degrees)**.** Let $a(\boldsymbol{x}) \in \Bbbk[x^1,\ldots,x^d]$ be a monomial, $a(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{n}}} = (x^1)^{n_1}\cdots(x^d)^{n_d}$. The *total degree* of $a(\boldsymbol{x})$ is $\deg(a) = n_1 + \cdots + n_d$; the *degree of* $a(\boldsymbol{x})$ *in coördinate* x^i is $\deg_i(a) = n_i$. The definition of total degree extends to an homogeneous polynomial, that is, to the sum of monomials if they all have the same total degree.

The *total degree shift* produced by the *complete*[2] generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$ of differential order k over dimension d is the integer $kd\cdot N/(d+1)$ and the *degree shift in coordinate* x^i is the integer[3] $kN/(d+1)$.

Corollary 8 (degree shift)**.** With the same notation as in Theorem 7, the complete generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$ acts on monomials $\boldsymbol{x}^{\vec{\boldsymbol{k}}_j}$ (with all degrees $k^i_j \in \mathbb{N}_{\geqslant 0}$) in such a way that the degrees of monomials satisfy the two relations:

$$\deg W^k_d(\boldsymbol{x}^{\vec{\boldsymbol{k}}_1},\ldots,\boldsymbol{x}^{\vec{\boldsymbol{k}}_N}) = \begin{cases} \sum_{j=1}^N \vec{\boldsymbol{k}}_j - \frac{kd}{d+1}N & \text{if} \quad \det \mathrm{Van}^k_d(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) \neq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{30}$$

and

$$\deg_i W^k_d(\boldsymbol{x}^{\vec{\boldsymbol{k}}_1},\ldots,\boldsymbol{x}^{\vec{\boldsymbol{k}}_N}) = \begin{cases} \sum_{j=1}^N \vec{\boldsymbol{k}}_j - \frac{kN}{d+1} & \text{if} \quad \det \mathrm{Van}^k_d(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_N) \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

That is, the degree of the resulting monomial is shifted down by the degree shift.

Remark 7. The degree shift $kN/(d+1)$ in a coordinate x^i, produced by the complete generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$, is exactly equal to the sum of differential orders w.r.t x^i (namely, $\sum_{j=1}^N r^i_j$) of the operators from the rows ($j \in \{1,\ldots,N\}$) in the Wronskian determinant. Equivalently, this degree shift equals the sum of degrees in x^i from the standard monomials (i.e. monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ corresponding to the operator $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ in the jth row of the Wronskian). Note that, by construction, $\vec{\boldsymbol{r}}_1 = (0,\ldots,0)$, $\vec{\boldsymbol{r}}_2 = (1,0,\ldots,0)$, and so on until $\vec{\boldsymbol{r}}_N = (0,\ldots,0,k)$. Now it is readily seen why $W^k_d(\boldsymbol{x}^{\vec{\boldsymbol{r}}_1}/\vec{\boldsymbol{r}}_1!,\ldots,\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}/\vec{\boldsymbol{r}}_N!) = 1$, a degree-zero monomial in d variables.

Proposition 9. Each of the following conditions alone is sufficient for the complete generalised Vandermonde determinant of d-tuples $\boldsymbol{m}_1,\ldots,\boldsymbol{m}_N \in \mathbb{Q}^d$ to be zero:

(*i*) duplicate columns: $\boldsymbol{m}_i = \boldsymbol{m}_{i'}$ for $i' \neq i$;
(*ii*) all degrees in a coordinate coincide: $m^i_1 = m^i_2 = \cdots = m^i_N$ for some $i \in \{1,\ldots,d\}$;
(*iii*) (*'deficient degree'*): if the power d-tuples of monomials $\boldsymbol{m}_1,\ldots,\boldsymbol{m}_N \in \mathbb{N}^d_{\geqslant 0}$ are such that their sum is less than the degree shift: $\sum_{j=1}^n m^i_j < kN/(d+1)$ for some i.

In all cases the determinant vanishing is equivalent to the columns being linearly dependent.

Proof. (*i*) Is immediate. (*ii*) Follows by subtracting m^i_1 times the row $(1,\ldots,1)$ of units from the $(i+1)$st row in the complete generalised Vandermonde determinant, obtaining an all-zeroes row. (*iii*) Follows from Theorem 7 as we have a deficiency of degree, which would produce a non-monomial – impossible. □

Remark 8 (multivariate generalisation of the Witt algebra). In Remark 2 on p. 25 (see also Theorem 3) we found the strong homotopy deformations of the Witt algebra in the one-dimensional case when N is even; this was achieved by applying a degree shift to the complex Laurent monomials (with possibly half-integer degrees). Upon such a shift the canonical

[2]Breaking the completeness assumption voids this definition. Indeed, the degree shift will be different in each coordinate; the total degree shift will be reduced by the total differential orders of whatever differential operators are excluded from the Wronskian (cf. [8] for the definition and discussion of incomplete Wronskians).

[3]Cf. Lemma 6: the shift $kN/(d+1)$ can be expressed as a binomial coefficient.

commutation relations were obtained: $[a_{i_1}, ..., a_{i_N}]_N = \Omega(i_1, ..., i_N) \cdot a_{i_1+\cdots+i_N}$ with $\Omega = \det \mathrm{Van}^{0,1,...,N-1}_{d=1}$ totally antisymmetric w.r.t its arguments.

We now continue with the deformation approach, generalising the Witt algebra over dimension $d > 1$. Take some generalised monomials $\boldsymbol{x}^{\vec{\boldsymbol{k}}_j} = (x^1)^{k^1_j} \cdots (x^d)^{k^d_j}$ with the rational power d-tuples $k^i_j \in \mathbb{Q}$ and coordinates $\boldsymbol{x} = (x^1, ..., x^d)$ over base dimension d; by means of Theorem 7, calculate the N-ary bracket $[\cdot, ..., \cdot]_N$, $N = \binom{d+k}{d}$, given by the complete generalised Wronskian determinant W^k_d of generalised monomial arguments. The result is a (generalised) monomial; its degree is made of up the component-wise sum of arguments' degrees minus the homogeneous shift $kN/(d+1)$. Apply the shift $s \in \mathbb{Q}$ to all arguments: $a_{\boldsymbol{i}_n} := \boldsymbol{x}^{\vec{\boldsymbol{i}}_n + s\mathbf{1}}$, where $\boldsymbol{i}_n \in \mathbb{Q}^d$ is the power d-tuple and $\mathbf{1} = (1, ..., 1)$. To recover the canonical commutation relations the shift must satisfy

$$Ns - \frac{kN}{d+1} = s, \qquad \text{hence} \qquad s = \frac{k}{d+1} \cdot \frac{N}{N-1}, \tag{32}$$

which in general is not an integer.[4] This gives the commutation relations

$$[a_{\boldsymbol{i}_1}, ..., a_{\boldsymbol{i}_N}]_N = \Omega(\boldsymbol{i}_1, ..., \boldsymbol{i}_N)\, a_{\boldsymbol{i}_1+\cdots+\boldsymbol{i}_N}, \tag{33}$$

where the structure constant $\Omega = \det \mathrm{Van}^k_d$ is totally antisymmetric w.r.t its arguments and the index sum is taken component-wise: $i^i_1 + \cdots + i^i_N$ with $i \in \{1, ..., d\}$.

2. Explicit construction of a class of polynomial algebras $\mathcal{A} \subseteq \Bbbk[x^1, \ldots, x^d]$ with Wronskian brackets

In this section we find all the finite-dimensional N-ary strongly homotopy (SH) Lie polynomial subalgebras $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$ containing the constant unit polynomial 1 both in the algebra, $1 \in \mathcal{A}$, and in the span of the image of the bracket, $1 \in \mathrm{span}_{\Bbbk} W^k_d(\mathcal{A}, ..., \mathcal{A})$. We are interested in this condition because if $1 \in \mathcal{A}$ is not in the (span of the) image of the bracket, the algebra $\mathcal{A}$ is not perfect (see Definition 2 on p. 22), hence it cannot be semi-simple; we shall give an equivalent condition for $1 \in \mathrm{span}_{\Bbbk} W^k_d(\mathcal{A}, ..., \mathcal{A})$.

We shall describe (a class of) all finite-dimensional algebras $\mathcal{A}$; we show that there are no others subject to the restrictions above. Among these finite-dimensional algebras, we characterise which are or cannot be perfect for each base dimension and differential order k; we give remedying examples of perfect algebras. Finally, we show that, subject to the assumptions above, no other finite-dimensional algebras exist, and we comment on why, even with the assumptions dropped, probably there exist no other (non-trivial, or perfect) finite-dimensional algebras.

Recall (cf. Remark 7 on p. 32) that $W^k_d(\boldsymbol{x}^{\vec{\boldsymbol{r}}_1}/\vec{\boldsymbol{r}}_1!, ..., \boldsymbol{x}^{\vec{\boldsymbol{r}}_N}/\vec{\boldsymbol{r}}_N!) = 1$, where $\vec{\boldsymbol{r}}_j$ is the power d-tuple of the differential operator $\partial^{\vec{\boldsymbol{r}}_j}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ in the jth row of the complete generalised Wronskian determinant $W^k_d = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}} = \bigwedge_{j=1}^N \partial^{\vec{\boldsymbol{r}}_j}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$. Replacing any argument by a higher total-degree monomial makes the degree sum strictly greater than $kN/(d+1)$, which is the shift downwards by the Wronskian determinant; hence 1 cannot be obtained on the new set of arguments. Let us formalise this claim and prove it (see Appendix B on p. 57).

[4] We remark that this shift is the same as the homogenisation constant y^* for homogenising the recurrence $y_n = y_{n-1} + y_{n-2} + \cdots + y_{n-N} - kN/(d+1)$. Namely, with $x_n = y_n - y^*$ we have $x_n = x_{n-1} + \cdots + x_{n-N}$.

Lemma 10 (consistency)**.** Fix any dimension $d \geqslant 1$ and any differential order $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let $a_1(\boldsymbol{x}), \ldots, a_N(\boldsymbol{x}) \in \Bbbk[\boldsymbol{x}]$ be arbitrary *monomials*. Then

$$W_d^k(a_1(\boldsymbol{x}), \ldots, a_N(\boldsymbol{x})) = 1 \qquad \text{implies} \qquad a_j(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j! \quad \forall j \leqslant N \tag{34}$$

up to (positive-sign) reordering and normalisation constants of a_j's such that their product equals $+1$.

We therefore immediately see that the requirement $1 \in \operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A}) \subseteq \mathcal{A}$ is equivalent to $\mathcal{A}$ containing all polynomials up to degree k, i.e. $\Bbbk_k[\boldsymbol{x}] \subseteq \mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$. The case $\mathcal{A} = \Bbbk_k[\boldsymbol{x}]$ is trivially closed as the resultant always lies in $\Bbbk$. Hence the following definition.

Definition 8 (Types of polynomial N-ary SH-Lie algebras)**.** Fix the dimension $d \geqslant 1$ and differential order $k \geqslant 1$; set $N = \binom{d+k}{k}$ and consider the complete generalised Wronskian determinant W_d^k. An SH-Lie subalgebra of polynomials $\mathcal{A} \subseteq \Bbbk[x^1, \ldots, x^d]$ with the N-ary bracket given by W_d^k is called *trivial* if $\mathcal{A} \subseteq \Bbbk_k[\boldsymbol{x}]$, and *consistent* (with respect to W_d^k) if it contains $\Bbbk_k[\boldsymbol{x}]$ as a proper subset, i.e. $\Bbbk_k[\boldsymbol{x}] \subsetneq \mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$.

In what follows we will usually take $\Bbbk_k[\boldsymbol{x}] \subsetneq \mathcal{A} \subseteq \Bbbk_{k+1}[\boldsymbol{x}]$. If $\mathcal{A}$ from $\Bbbk_{k+1}[\boldsymbol{x}]$ contains (the span of) exactly one $(k+1)$th degree monomial or homogeneous polynomial, i.e. $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p \rangle$, we say that $\mathcal{A}$ is *lonely*.[5] On the other hand, if $\mathcal{A}$ contains at least two linearly independent $(k+1)$th degree monomials or homogeneous polynomials $p \not\sim q \in \Bbbk_{k+1}[\boldsymbol{x}]$, that is, $\mathcal{A} \supseteq \Bbbk_k[\boldsymbol{x}] \oplus \langle p, q \rangle$, we say that the algebra $\mathcal{A}$ is *chubby*.[6]

We say that the algebra $\mathcal{A}$ is *tall* if it contains any polynomial of degree $> k+1$. If a tall algebra $\mathcal{A}$ contains only one monomial of degree $k+1$, then we say that $\mathcal{A}$ is *lanky*; we will soon see that in this case, $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle (x^i)^{k+1}, \ldots, (x^i)^{k+\ell} \rangle$ for some fixed $i \in \{1, \ldots, d\}$ and $\ell \geqslant 2$ (see Lemma 11 below). If, on the contrary, a tall algebra contains multiple linearly independent monomials (or homogeneous polynomials) of degree $k+1$, then the tall algebra $\mathcal{A}$ is also chubby.

Using the fact that the algebra $\mathcal{A}$ is closed under the Wronskian determinant, we will produce the $(k+1)$th degree monomials as resulting from the Wronskian determinant by using arguments from $\mathcal{A}$. By multilinearity and the degree-shift mechanism in Theorem 5 on p. 30, it suffices to consider $\mathcal{A}$ spanned by monomials. The following lemma explains our choice of terminology in Definition 8 (see also the visualisation in Figure 1).

Lemma 11. Suppose $\mathcal{A}$ is a tall polynomial SH-Lie algebra, i.e. $\mathcal{A}$ contains a monomial[7] $a(\boldsymbol{x})$ of degree $\deg(a) > k+1$, and suppose $\mathcal{A}$ is consistent: now $\mathcal{A} \supseteq \Bbbk_k[\boldsymbol{x}] \oplus \langle a(\boldsymbol{x}) \rangle$. Depending on $a(\boldsymbol{x})$ we have the following:

(*i*) if $a(\boldsymbol{x}) = (x^i)^{k+\ell}$ for some i with $\ell \geq 2$, then $(x^i)^{k+\ell-1}, \ldots, (x^i)^{k+1} \in \mathcal{A}$, (*lanky* $\mathcal{A}$)

(*ii*) otherwise (i.e. $x^i x^j | a(\boldsymbol{x})$ for some $j \neq i$) the algebra $\mathcal{A}$ is chubby. (*chubby* $\mathcal{A}$)

We postpone the proof until we will have proven a structure Theorem 13 (needed for the proof). See §2.3 for the proof on p. 42.

[5] This class, $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p(\boldsymbol{x}) \rangle$, will form the class of finite-dimensional algebras (see Theorem 14 on p. 38); it is this what permits us to write the equality sign $=$ in the definition: 'Lonely is finite-dimensional.'

[6] We will show in Theorem 17 on p. 50 that none such algebra is finite-dimensional: 'Chubby is infinite-dimensional.'

[7] More generally, for a polynomial $p(\boldsymbol{x})$ in place of $a(\boldsymbol{x})$, extend the results of the lemma by decomposing $p(\boldsymbol{x})$ into its constituent monomials and applying the lemma to each monomial.

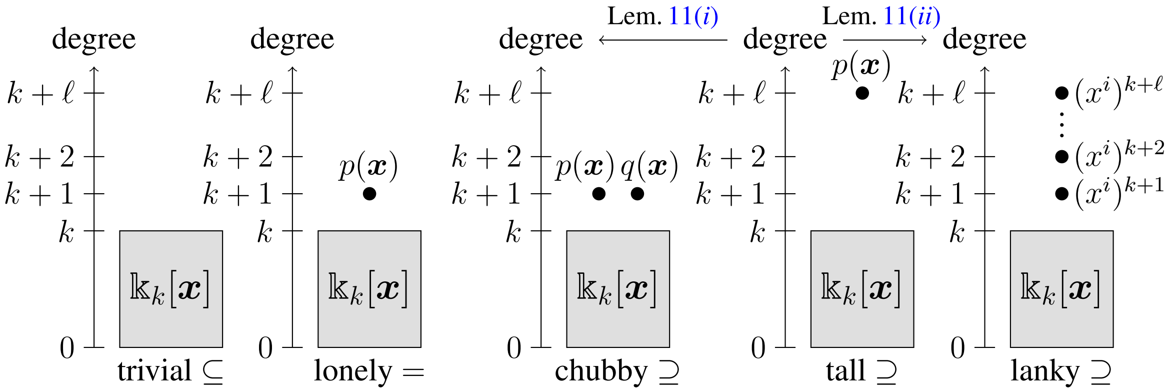

FIGURE 1. Visualisation of polynomial SH-Lie algebras from Definition 8. The set inclusions denote whether the subspace of polynomials from $\Bbbk[\boldsymbol{x}]$ drawn is a subset of the algebra ($\subseteq$), equal to it ($=$), or contained in it ($\supsetneq$).

2.1. **Construction of finite-dimensional subalgebras and their structure constants.** In this (sub)section we find all consistent (i.e. $\mathcal{A} \supsetneq \Bbbk_k[\boldsymbol{x}]$ or, equivalently, $1 \in \mathcal{A}$ such that also $1 \in \operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A})$) finite-dimensional polynomial subalgebras $\mathcal{A} \subsetneq \Bbbk[\boldsymbol{x}]$ over any base dimension $d \geqslant 1$ and any differential order $k \geqslant 1$ in the complete generalised Wronskian determinant as the SH-Lie N-ary bracket $[\cdot, \ldots, \cdot]_N = W_{d \geqslant 1}^{k \geqslant 1} = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$.

We now give a special case of the first main theorem: existence of a finite-dimensional class of polynomial algebras $\mathcal{A}$. Namely, suppose $\mathcal{A}$ is lonely and we restrict to the special case when p of $\deg(p) = k+1$ is a *monomial*; we denote the monomial by $a(\boldsymbol{x})$. To show that any homogeneous p of $\deg(p)$ works too, we need a structure Theorem 13.

Lemma 12 (special case of our first main Theorem 14: existence)**.** Fix any dimension $d \geqslant 1$ and any order $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle a(\boldsymbol{x}) \rangle$, where $a(\boldsymbol{x})$ is a *monomial* of $\deg(a) = k+1$. Then $\mathcal{A}$ is *closed* under the bracket $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$. In particular, $\mathcal{A}$ is a finite-dimensional subalgebra of $\Bbbk[\boldsymbol{x}]$ with the Wronskian W_d^k as the N-ary bracket.

Proof. If all arguments in the Wronskian determinant are of order at most k, the result is a constant monomial; the sum of degrees in all N monomials in $\Bbbk_k[\boldsymbol{x}]$ is thus $kN/(d+1)$. Replacing the argument $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j! \in \Bbbk_k[\boldsymbol{x}]$ (see Notation 4) by the monomial $a(\boldsymbol{x})$ of degree $\deg(a) = k+1$ yields, via the Wronskian determinant W_d^k, the monomial of which the degree in coordinate x^i is:

$$
\begin{aligned}
\deg_i W_d^k(1, \boldsymbol{x}^{\vec{\boldsymbol{r}}_2}/\vec{\boldsymbol{r}}_2!, \ldots, \underbrace{a(\boldsymbol{x})}_{\widehat{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!}}, \ldots, \boldsymbol{x}^{\vec{\boldsymbol{r}}_N}/\vec{\boldsymbol{r}}_N!) &= \frac{kN}{d+1} - \deg_i(\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}) + \deg_i a(\boldsymbol{x}) - \frac{kN}{d+1} \\
&= \deg_i a(\boldsymbol{x}) - \deg_i(\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}) \leq \deg_i a(\boldsymbol{x}),
\end{aligned}
$$

where equality is attained only if $\deg_i(\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}) = 0$. The total degree of the resulting monomial can therefore be $k+1$ only if $\deg_i(\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}) = 0$ for every $i \in \{1, \ldots, d\}$, which means that $j = 1$ and $\boldsymbol{x}^{\vec{\boldsymbol{r}}_1} = 1$; this yields the monomial $a(\boldsymbol{x})$ itself (up to a constant). In particular, no other monomials of degrees $k+1$ (or higher) can be obtained. The subspace $\mathcal{A}$ is thus closed under W_d^k. This completes the proof. $\square$

Example 9. Let the base dimension be $d = 3$ and the differential order be $k = 1$; here $\Bbbk_k[\boldsymbol{x}] = \langle 1, x, y, z\rangle$. The polynomial subspaces

$$\mathcal{A} = \langle 1, x, y, z, x^2\rangle, \quad \text{or,} \quad \mathcal{A} = \langle 1, x, y, z, xy\rangle, \quad \ldots, \quad \text{or,} \quad \mathcal{A} = \langle 1, x, y, z, z^2\rangle, \tag{35}$$

are all closed under the Wronskian determinant $W^{k=1}_{d=3} = \mathbf{1} \wedge \partial/\partial x \wedge \partial/\partial y \wedge \partial/\partial z$. For instance, setting $\mathcal{A} = \langle 1, x, y, z, xy\rangle$, we have

$$W^{k=1}_{d=3}(x, y, z, xy) = \det \begin{vmatrix} x & y & z & xy \\ 1 & 0 & 0 & y \\ & 1 & 0 & x \\ & & 1 & 0 \end{vmatrix} = xy,$$

calculated explicitly, and

$$W^{k=1}_{d=3}(1, y, z, xy) = \det \begin{vmatrix} 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 \\ & 1 & 0 & 1 \\ & & 1 & 0 \end{vmatrix} y = y,$$

calculated by formula (24) in Theorem 7 on p. 30. Notice that the monomial z is not in the image of the bracket, $z \notin \operatorname{span}_{\Bbbk} W^k_d(\mathcal{A}, \ldots, \mathcal{A})$, as any choice of independent monomial arguments from $\mathcal{A} = \langle 1, x, y, z, xy\rangle$ can have at most sum-of-degrees in z be $1 = kN/(d+1)$.

Example 10. Take both the dimension and differential order to be $d = 2 = k$. Then we obtain finite-dimensional polynomial algebras $\mathcal{A} = \langle 1, x, y, x^2, xy, y^2\rangle \oplus \langle a(\boldsymbol{x})\rangle$ for any of the choices $a(\boldsymbol{x}) = x^3$, x^2y, xy^2, y^3. For instance, with $a(\boldsymbol{x}) = xy^2$:

$$W^{k=2}_{d=2}(1, x, y, x^2, xy, xy^2) = 4x,$$

where we replaced the argument y^2 by xy^2. Note that there is no way to obtain the monomial x^2 as the result of the Wronskian determinant with arguments from this $\mathcal{A} = \Bbbk_2[\boldsymbol{x}] \oplus \langle xy^2\rangle$: the arguments' degree sum minus shift in $x^1 = x$ is at most $\deg_x a(\boldsymbol{x}) = 1$. The algebra $\mathcal{A}$ is hence not perfect (see Definition 2 on p. 22 and Proposition 15 on p. 39).

Example 11 (appending Laurent monomials)**.** Let us extend the finite-dimensional ternary polynomial algebra $\mathcal{A} = \langle 1, x, y, xy\rangle$ over base dimension $d = 2$ by the span of Laurent monomials: $\mathcal{A}' = \mathcal{A} \oplus \mathcal{L}$, where $\mathcal{L}$ is generated by pure Laurent monomials and monomials with degree at most 1:

$$\mathcal{L} = \langle\{\boldsymbol{x}^{\vec{\boldsymbol{n}}} : \vec{\boldsymbol{n}} \in \mathbb{Z}^d_{\leqslant 1}\}\rangle = \langle x/y, x/y^2, \ldots, y/x, y/x^2, \ldots, 1/x, 1/y, 1/xy, 1/xy^2, \ldots\rangle. \tag{36}$$

It is clear that $\mathcal{L}$ alone is closed under the ternary bracket: the only way to get a monomial with degree > 1 in a coordinate is for all arguments from $\mathcal{L}$ to have that coordinate to degree 1; this makes the second row in the complete generalised Vandermonde determinant equal to the first, hence the determinant vanishes (see Proposition 9 *(ii)* on p. 9).

Setting $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$, we have the following ternary commutation relations:

$$[1, x, xy] = x, \qquad [1, y, xy] = -y, \qquad [x, y, xy] = -xy,$$

describing the finite-dimensional algebra itself, and

$$[x, y, 1/y] = 2/y, \qquad [xy, 1/x, 1/y] = 3/xy \qquad [1, x, /y] = -1/y^2,$$

describing the interaction with the purely-negative degrees, and

$$[x, xy, 1/y] = x/y, \qquad [x/y, xy, 1/y] = 2x/y^4 \qquad [x, x/y, x/y^4] = 0,$$

which shows that $\mathcal{A}' = \mathcal{A} \oplus \mathcal{L}$ is closed. Note also that $[y, xy, x/y] = -2x \in \mathcal{A}$.

The proof of Lemma 12 is silent about the coefficient – our generalised Vandermonde determinant. In particular, can we take an homogeneous polynomial p of total degree $\deg(p) = k+1$? We now give a structure theorem: an explicit factorisation of the complete generalised Vandermonde determinant[8] relevant for our case. Indeed, from $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p \rangle$ the only choices of N monomial arguments are the divided power monomials (all from $\Bbbk_k[\boldsymbol{x}]$) or one such monomial replaced by p.

Theorem 13 (*structure of loneliness*: structure constants for lonely algebras $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle a \rangle$). *Fix any dimension $d \geqslant 1$ and any differential order $k \geqslant 1$, and set $N = \binom{d+k}{k}$. For any fixed index $j \in \{1, ..., N\}$ and any* <u>*monomial*</u> *$a(\boldsymbol{x}) = (x^1)^{n_1} \cdots (x^d)^{n_d}$ we have*

$$W_d^k\Big(1, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, ..., \underbrace{a(\boldsymbol{x})}_{\widehat{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!}}, ..., \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!}\Big) = \frac{(-)^{k-r_j}}{(k-r_j)!} \prod_{i=1}^{d} \left\{ \prod_{\ell=0}^{r_j^i - 1} (n_i - \ell) \right\} \prod_{\ell=r_j+1}^{k} (\deg a - \ell) \cdot \frac{a(\boldsymbol{x})}{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}}, \tag{37}$$

where if $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j} \nmid a(\boldsymbol{x})$ then the first product in the coefficient is zero.

Remark 9. With the same notation as in Theorem 13 above, we rewrite the structure constant in terms of a differential. Namely, by Eq. (12) on p. 24 the first product in (37) equals the coefficient of the differential operator $\partial^{|\vec{\boldsymbol{r}}_j|}/\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ acting on $a(\boldsymbol{x})$, that is

$$W_d^k\Big(1, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, ..., \underbrace{a(\boldsymbol{x})}_{\widehat{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!}}, ..., \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!}\Big) = \frac{(-)^{k-r_j}}{(k-r_j)!} \prod_{\ell=r_j+1}^{k} (\deg a - \ell) \cdot \frac{\partial^{|\vec{\boldsymbol{r}}_j|}}{\partial \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}} a(\boldsymbol{x}), \tag{38}$$

i.e. the missing term differentiates the term that replaced it.

Remark 10. This structure theorem permits easy computation of the Wronskian bracket for any arguments in a lonely polynomial algebra $\mathcal{A}$.

Proof (of Theorem 13). By Theorem 7 about complete generalised Vandermonde determinants on p. 30 we immediately obtain the monomial term $a(\boldsymbol{x})/\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$; recall that whenever $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ does *not* divide the monomial $a(\boldsymbol{x})$, the overall constant equals zero, hence the division by higher degrees (which would result in negative degrees) is fictitious (see Eq. (12) on p. 24 and the surrounding commentary). It remains to work out that constant – by simplifying the generalised Vandermonde determinant. Note that the determinant contains $\boldsymbol{n} = (n_1, ..., n_d)$ values only in one column; their products go up to at most total degree k in them. We view the determinant as a polynomial $g \in \Bbbk[n_1, ..., n_d]$; the polynomial can have at most k factors linear in n_i's (if linear factors exist).

[8]The complete generalised Vandermonde determinant involved (even in the triangular form) is difficult to expand; it is possible in $k = 1$. See Appendix A for the direct proof.

Split this constant into a part that depends on $\boldsymbol{n}$ multiplied by an overall constant independent of the degrees of coordinates, $\boldsymbol{n}$, of $a(\boldsymbol{x})$:

$$W_d^k\Big(1, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, \ldots, \underbrace{a(\boldsymbol{x})}_{\widehat{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!}}, \ldots, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!}\Big) = (\text{const. indep. of } \boldsymbol{n}) \cdot (\text{number dep. on } \boldsymbol{n}) \cdot \frac{a(\boldsymbol{x})}{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}}. \quad (39)$$

We work out the $\boldsymbol{n}$-dependent part first.

Note that taking $a(\boldsymbol{x}) \sim \boldsymbol{x}^{\vec{\boldsymbol{r}}_{j'}}$ for some $j' \neq j$ we will obtain the zero polynomial. Indeed, for any choice of monomial a of degree $r_j < \deg(a) \leq k$, the monomial a must hit another monomial, as the set of arguments is complete up to degree k. This gives $k - r_j$ factors of the form $(\deg a - \ell)$ for ℓ values $\ell \in \{r_j + 1, \ldots, k\}$, hence their product $\prod_{\ell=r_j+1}^{k}(\deg p - \ell)$ divides the monomial g; we let f be the quotient obtained by division.

As $\deg_{\boldsymbol{n}=(n_1,\ldots,n_d)} g = k$ and the product above is of degree $k - r_j$, we find that $\deg_{\boldsymbol{n}} f = r_j$. Let $i \in \{1, .., d\}$ be arbitrary. Then for the choices $1, x^i, (x^i)^2, \ldots, (x^i)^{r_j^i}$ of $a(\boldsymbol{x})$ we obtain a repeated argument. This gives the linear factors $n_i - \ell$ for $\ell \in \{0, \ldots, r_j^i\}$. We thus obtain the first product in (37). Inspecting the degree we find $r_j^i + \cdots + r_j^d = r_j$, therefore we have found all the linear ($\boldsymbol{n}$-dependent) factors (hence the $\boldsymbol{n}$-dependent part in Eq. (39)).

All that is left is to determine the overall constant (independent of $\boldsymbol{n}$). For this we use that the choice $p(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}$ gives $W_d^k(\ldots, p(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{r}}_j}, \ldots) = \vec{\boldsymbol{r}}_j!$. The first product in (37) yields $\vec{\boldsymbol{r}}_j!$, whereas the second yields $(-)^{k-r_j}(k - r_j)!$. This completes the proof. □

We now state our first main theorem (recall the special case in Lemma 12).

Theorem 14 (main theorem **1**: lonely algebras are finite-dimensional)**.** *Fix any dimension $d \geqslant 1$ and any order $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let the polynomial subspace $\mathcal{A}$ be $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p \rangle$, where p is any homogeneous polynomial of total degree $\deg(p) = k + 1$. Then the subspace $\mathcal{A}$ is closed under the bracket given by $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$. In conclusion, $\mathcal{A}$ is a finite-dimensional N-ary SH-Lie subalgebra of $\Bbbk[\boldsymbol{x}]$ with the complete generalised Wronskian determinant as the N-ary bracket.*

Proof. By Theorems 7 and 13, it remains only to consider the bracket value $W_d^k(p(\boldsymbol{x}), \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, \ldots, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!})$. Split $p(\boldsymbol{x})$ into its constituent sum of monomials in $\Bbbk_{k+1}[\boldsymbol{x}]$ (not necessarily belonging to $\mathcal{A}$); by multilinearity we consider each term separately. By Theorem 5 (or 13) the resulting monomial from each monomial argument is proportional to the monomial itself; it remains to show that each resulting monomial is preceded by the same overall constant. From formula (37) with $\vec{\boldsymbol{r}}_1 = (0, \ldots, 0)$, so $r_1 = 0$, and as the degree of each constituent monomial in $p(\boldsymbol{x})$ is equal to $k + 1$, we have that each monomial is preceded by the factor $(-)^k/k! \cdot \prod \left\{\prod_{\varnothing}\right\} \cdot (k+1-1)(k+1-2)\cdots(k+1-k) = (-)^k$, which is independent of the monomial. Hence the resulting polynomial is proportional to $p(\boldsymbol{x})$ itself, as required. □

2.2. **(Im)perfect N-ary SH-Lie polynomial algebras.** Any (semi-)simple Lie algebra is perfect. The definition of perfect Lie algebras extends to the N-ary case (see Definition 2 on p. 22) – the (span of the) N-ary bracket is surjective: $\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A}) = \mathcal{A}$. In this section we characterise which of the finite dimensional N-ary SH-Lie algebras found in Lemma 12 on p. 35 and the first main Theorem 14 are perfect: we show that over arbitrary base dimension $d \geqslant 1$ and differential order $k \geqslant 1$, modulo a list of exceptions, a *monomial* as the top-degree polynomial (that is, a monomial $a(\boldsymbol{x})$ of degree $\deg(p) = k + 1$ in the lonely algebra

$\mathcal{A} = \Bbbk_k \boldsymbol{x} \oplus \langle a(\boldsymbol{x}) \rangle)$ is *not* enough for the algebra to be perfect. We give an example of a perfect finite-dimensional algebra with an homogeneous polynomial on top. Determining which homogeneous polynomial as the top-degree polynomial results in a perfect (lonely) N-ary SH-Lie polynomial algebra remains an open problem.

Proposition 15 (monomial imperfection)**.** Let $\mathcal{A}$ be a consistent lonely N-ary SH-Lie polynomial subalgebra, i.e. $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle a(\boldsymbol{x}) \rangle$, where $a(\boldsymbol{x}) = (x^1)^{n_1} \cdots (x^d)^{n_d}$ is a *monomial*[9] of degree $\deg(a) = n_1 + \cdots + n_d = k + 1$. Then the algebra $\mathcal{A}$ is *not perfect*, i.e. the image of the bracket (given by the complete generalised Wronskian of differential order k over base dimension d) does not span the algebra, $\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A}) \neq \mathcal{A}$, unless either of the following cases is satisfied:

$$(i)\ d = 1 \text{ (and any } k \geqslant 1 \text{ value)}, \qquad \text{or} \qquad (ii)\ d = 2 \text{ and } k = 1;$$

i.e. every lonely (thus finite-dimensional) algebra in dimension one is perfect, whereas over multi-dimension there exists only one example of a perfect algebra with a monomial on top.

Note that the exception (*i*) corresponds to the class of perfect finite-dimensional algebras found in Example 3, and (*ii*) to Example 3 on p. 21.

Remark 11. Over dimension $d = 1$ with coordinate x there is only one consistent finite-dimensional N-ary polynomial SH-Lie algebra $\mathcal{A} = \langle 1, x/1!, x^2/2!, \ldots, x^N/N! \rangle$ with the ordinary Wronskian determinant $W_{d=1}^{0,1,\ldots,N-1}$ bracket of differential order $N-1$ and N arguments; this algebra is perfect, $\operatorname{span}_{\Bbbk} W_{d=1}^{0,1,\ldots,N-1}(\mathcal{A}, \ldots, \mathcal{A}) = \mathcal{A}$, and, according to the given Definition 8, lonely.

Proof (of Proposition 15). By Theorem 7, taking brackets of monomial arguments yields monomials; by multilinearity it then suffices to consider the span of brackets of monomial arguments. Supposing the algebra $\mathcal{A}$ is perfect, we must have the monomial $(x^i)^k$ in the image of the Wronskian determinant. By Theorem 7, the only way to obtain this monomial is through the choice of argument $a(\boldsymbol{x})$ such that $(x^i)^k = a(\boldsymbol{x})/x^\ell$ for some $\ell \in \{1, \ldots, d\}$. Then $(x^i)^k | a(\boldsymbol{x})$ for every $i \in \{1, \ldots, d\}$, which implies that the least common multiple $(x^1)^k \cdots (x^d)^k$ divides $p(\boldsymbol{x})$. It follows that $kd \leq k + 1$, or, equivalently, $k(d-1) \leq 1$. The only solutions of $k, d \in \mathbb{N}_{\geqslant 1}$ that satisfy the requirement are (*i*) and (*ii*). $\square$

Remark 12 (shrinking of the algebra by iterating the bracket[10]). Let the base dimension be $d \geqslant 2$ and put the Wronskian W_d^k as the N-ary bracket. Take a *monomial* $a(\boldsymbol{x})$ as the $(k+1)$th degree polynomial in a lonely algebra: $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle a(\boldsymbol{x}) \rangle$, hence $\dim_{\Bbbk} \mathcal{A} = N + 1$. Suppose the SH-Lie algebra $\mathcal{A}$ is *not perfect*. Upon applying the bracket, we will lose some monomials as arguments:

$$\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A}) \subsetneq \mathcal{A}, \qquad \text{hence} \qquad \dim \operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, \ldots, \mathcal{A}) \leqslant N.$$

If at least two monomials are lost, we no longer have enough linearly-independent arguments to obtain a non-vanishing bracket. Supposing only one monomial argument is lost in the span, we now have only one set of N linearly-independent arguments, hence re-applying the bracket yields only one monomial up to constant (possibly zero).

[9]This assumption is essential. In Example 12 homogeneous polynomials will alleviate this imperfection.

[10]The authors thank Maxim Kontsevich for this question, asked during the talk by the second author at the Mathematics Seminar, IHÉS. In this remark we develop the question more precisely and answer it.

Therefore only one or two iterated applications of the brackets are possible before the bracket of any arguments will vanish identically: the dimension decreases as

$$N+1 \to N \to 1 \to 0 \quad \text{or} \quad N+1 \to N \to 0, \quad \text{or as} \quad N+1 \to (<N) \to 0. \tag{40}$$

The problem is to determine for which choices of $a(\boldsymbol{x})$, which case in Eq. (40) is obtained?

Let us now answer this question by considering when the first two cases in (40) can be satisfied. Suppose the degree of x^i in $a(\boldsymbol{x})$ is less than k; it is then not possible to obtain the monomial $(x^i)^k$ as the result of the bracket. In the next iteration, any choice of N independent monomials from $\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A}, ..., \mathcal{A})$ as arguments will then be deficient in x^i (see Definition 10 on p. 53 and Proposition 9 *(iii)* on p. 32), hence all brackets will vanish. The first case is thus *impossible*. If in the first iteration only one monomial is lost, it must be $(x^i)^k$, as losing any other immediately implies that more are lost (increase the degree of any coordinate while decreasing another). The dimension must be $d = 2$, as otherwise more than one monomial of the form $(x^i)^k$ will be lost; the differential order cannot exceed $k = 2$ as for $k \geqslant 3$, setting $a(\boldsymbol{x}) = x^k y$, we lose $y^2, ..., y^k$.

Therefore the cases in (40) are satisfied only by:

$$N+1 \to N \to 1 \to 0 \qquad \text{impossible}, \tag{41}$$

$$\begin{aligned} N+1 \to N \to 0 \qquad & (d=2=k): \quad a(\boldsymbol{x}) = x^2y \text{ or } xy^2, \text{ and} \\ & (d=2, k=1): \quad a(\boldsymbol{x}) = x^2 \text{ or } y^2, \end{aligned} \tag{42}$$

$$N+1 \to (<N) \to 0 \qquad \text{every other choice of } a(\boldsymbol{x}). \tag{43}$$

This solves the problem of decreasing imperfect algebras *of monomials* by iterated brackets. Among the monomials in case (43), it is a problem of combinators to determine to what value the dimension shrinks in each case: in each base variable the resulting monomial has (non-negative) degree not exceeding that of $a(\boldsymbol{x}) = (x^1)^{n_1} \cdots (x^d)^{n_d}$; the number of such monomials is $\prod_{i=1}^d (n_i + 1)$. Case (42) corresponds precisely to when this product equals N.

Example 12 (perfection!)**.** In Proposition 15 the assumption that $a(\boldsymbol{x})$ is a monomial was *essential*. Let the base dimension be $d = 3$ and the differential order be $k = 1$. By Proposition 15 then, a 2nd degree monomial $a(\boldsymbol{x})$ as the top-degree polynomial is not enough to form a perfect algebra $\mathcal{A} = \langle 1, x, y, z, a(\boldsymbol{x})\rangle$. We now illustrate that an homogeneous polynomial of degree 2 can give a perfect algebra: put $p(\boldsymbol{x}) = xy + yz + zx$, which gives for $\mathcal{A} = \langle 1, x, y, z, xy + yz + zx\rangle$ the commutation relations $[x, y, z, p(\boldsymbol{x})] = p(\boldsymbol{x})$,

$$[1, x, y, p(\boldsymbol{x})] = x + y, \quad [1, x, z, p(\boldsymbol{x})] = -x - z, \quad \text{and} \quad [1, y, z, p(\boldsymbol{x})] = y + z, \tag{44}$$

from which $\langle W_{d=3}^{k=1}(\mathcal{A}, \mathcal{A}, \mathcal{A}, \mathcal{A})\rangle = \mathcal{A}$ follows as $x = ((x + y) + (-x - y) + (y + z))/2$ using which we also obtain the remaining monomials y and z.

Notice that each r.-h.s. in (44) contains exactly two monomial terms; the coordinate missing in the r.-h.s. was missing from the set of arguments in the l.-h.s. (cf. Proposition 9 on p. 32, condition *(iii)* on deficiency). For instance, the degree in z among $1, x, y, xy$ sum up to $0 < kN/(d + 1) = 1$. We will return to this point in the discussion (see Definition 10 on p. 53).

Open problem 1. Let the base dimension $d \geqslant 2$ and differential order $k \geqslant 1$ be arbitrary. Put the complete generalised Wronskian $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$ as the N-ary bracket, $N = \binom{d+k}{d}$, on the N-ary SH-Lie polynomial algebra $\Bbbk[x^1, ..., x^d]$. Does there exist an homogeneous polynomial $p(\boldsymbol{x}) \in \Bbbk[\boldsymbol{x}]$ of degree $k+1$ such that the N-ary SH-Lie polynomial subalgebra $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p(\boldsymbol{x})\rangle$ is *perfect* w.r.t the Wronskian, i.e. $\operatorname{span}_\Bbbk W_d^k(\mathcal{A}, ..., \mathcal{A}) = \mathcal{A}$?

Comment (to Open problem 1)**.** Fix $d \geqslant 2$ and $k \geqslant 1$ as in Problem 1. Build the homogeneous polynomial $p(\boldsymbol{x})$ from $(k+1)$th degree monomials; there are a total of

$$\binom{d + (k+1) - 1}{k+1} = \binom{d+k}{k+1} = \frac{d}{k+1} N \tag{45}$$

monomials of total degree $k+1$ (cf. Lemma 6 on p. 29, where the fraction in (45) is inverted). The dimension of the algebra $\mathcal{A}$ is $N+1$, where N monomials come from $\Bbbk_k[\boldsymbol{x}]$, of which one is the constant monomial; the polynomial $p(\boldsymbol{x})$ is always in the image (see Theorem 13). As we demanded that the algebra $\mathcal{A}$ is consistent, the unit constant polynomial is also in the image. Hence for the algebra to be perfect, all non-constant monomials up to degree k must be in the span of the (homogeneous[11]) $N-1$ polynomials $q_2(\boldsymbol{x}), ..., q_N(\boldsymbol{x})$ obtained by

$$q_j(\boldsymbol{x}) = W_d^k(1, \boldsymbol{x}^{\vec{\boldsymbol{r}}_2}/\vec{\boldsymbol{r}}_2!, ..., \widehat{\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j! \mapsto p(\boldsymbol{x})}, ..., \boldsymbol{x}^{\vec{\boldsymbol{r}}_N}/\vec{\boldsymbol{r}}_N!); \tag{46}$$

(see Theorem 13 and Remark 9 on p. 37 for the computation and Notations 1 and 2 on p. 23 for the power d-tuple notation). The problem is thus two-fold:

- (*i*) (combinatorial part) calculate how many and which constituent monomials make up each polynomial $q_j(\boldsymbol{x})$ (depending on j, so on $r_j = |\vec{\boldsymbol{r}}_j|$ and $\vec{\boldsymbol{r}}_j = (r_j^1, ..., r_j^d)$);
- (*ii*) (linear algebra part) determine whether the $N-1$ polynomials obtained this way are linearly independent (hence their span is $(N-1)$-dimensional as required).

Part *(ii)* can be reduced to computing an $(N-1) \times (N-1)$ determinant (of numbers).

2.3. **Non-existence of other (*consistent*) finite-dimensional SH-Lie polynomial algebras.** In this section we look beyond the class of *lonely* N-ary strongly homotopy (SH) Lie polynomial algebras $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p(\boldsymbol{x})\rangle$ (where $p(\boldsymbol{x})$ is an homogeneous polynomial of degree $\deg p = k+1$) studied in §2.1. We recall that every lonely algebra is closed w.r.t the complete generalised Wronskian determinant (of differential order k over dimension d) $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$, and hence finite-dimensional (see the first main Theorem 14 on p. 38 and Definition 8 on p. 34 for the relevant definitions). We now show that a consistent (i.e. $\Bbbk_k[\boldsymbol{x}] \subsetneq \mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$) algebra $\mathcal{A}$ is finite-dimensional if and *only if* $\mathcal{A}$ contains only a single homogeneous polynomial of degree $k+1$ (i.e $\mathcal{A}$ is lonely) and no other polynomial(s) of higher degree(s). In the end of this section we discuss whether we can drop the assumption that the algebra $\mathcal{A}$ is consistent (or, equivalently, by Lemma 10 that the constant unit polynomial 1 is both in the algebra $1 \in \mathcal{A}$ and the image of the bracket: $1 \in \operatorname{span}_\Bbbk W_d^k(\mathcal{A}, ..., \mathcal{A})$); we formulate an open problem about this.

Essentially, our goal is to produce monomials of higher degree from a collection of (poly- or) monomials in a consistent algebra using the complete generalised Wronskian determinant. To this end, let us consider algebras which are not precisely lonely; specifically, let us consider consistent algebras which contain a polynomial of too high a degree ($> k+1$) or too many (at least two) linearly independent polynomials of degree $k+1$; see Figure 1 on p. 35 for an illustration.

[11] The total degree of $q_j(\boldsymbol{x})$ is $\deg q_j(\boldsymbol{x}) = k + 1 - r_j$, where r_j is the total degree of the replaced monomial as argument.

Recall that Lemma 11 states that a tall algebra reduces to a chubby or a lanky algebra (see Definition 8 on p. 34) depending on the choice of top-degree polynomial $p(\boldsymbol{x})$. It suffices, by multilinearity of the Wronskian determinant, to let $p(\boldsymbol{x})$ be a monomial, $p(\boldsymbol{x}) = a(\boldsymbol{x})$.[12]

We now *re*-state and prove Lemma 11 (first stated on p. 34).

Lemma 11. Suppose $\mathcal{A}$ is a tall polynomial SH-Lie algebra, i.e. $\mathcal{A}$ contains a monomial[13] $a(\boldsymbol{x})$ of degree $\deg(a) > k + 1$, and suppose $\mathcal{A}$ is consistent: now $\mathcal{A} \supseteq \Bbbk_k[\boldsymbol{x}] \oplus \langle a(\boldsymbol{x})\rangle$. Depending on $a(\boldsymbol{x})$ we have the following:

(i) if $a(\boldsymbol{x}) = (x^i)^{k+\ell}$ for some i with $\ell \geq 2$, then $(x^i)^{k+\ell-1}, \ldots, (x^i)^{k+1} \in \mathcal{A}$, (*lanky* $\mathcal{A}$)

(ii) otherwise (i.e. $x^i x^j | a(\boldsymbol{x})$ for some $j \neq i$) the algebra $\mathcal{A}$ is chubby. (*chubby* $\mathcal{A}$)

Proof of Lemma 11. (i) As $\ell > 1$, then by the structure Theorem 13 on p. 37, taking the bracket with x^i as argument replaced by $a(\boldsymbol{x})$ yields the monomial $(x^i)^{k+\ell-1}$ of preceding degree times a non-zero constant. Repeating this descent $\ell \mapsto \ell - 1$ until $(x^i)^{k+1}$ is obtained verifies the claim.

(ii) Similarly to (i), but now there are at least two different ways to divide (at least by either x^i or x^j), obtaining non-proportional monomials. □

Unlike we had only one monomial $a(\boldsymbol{x})$ in the structure Theorem 13, let us now replace two standard monomials in the set of arguments for the Wronskian $W_{d\geqslant 1}^{k\geqslant 1}$. Specifically, let us replace $1 = \boldsymbol{x}^{\vec{r}_1}$ and $x^1 = \boldsymbol{x}^{\vec{r}_2}$ by *monomials* $q(\boldsymbol{x})$ and $p(\boldsymbol{x})$, respectively. The total degree of the resulting monomial from the Wronskian of the above arguments is at least $2k + 1 > k + 1$ if the coefficient in front is non-zero. The task is to factorise the complete generalised Vandermonde determinant; we do this in Theorem 16 on p. 44. The examples below suggest the general form, which we prove:

Example 13 ($k = 1$, $d = 2$, $\widehat{x} \mapsto p$)**.** Let the base dimension be $d = 2$ and consider the complete generalised Wronskian of differential order $k = 1$: $W_{d=2}^{k=1} = \mathbf{1} \wedge \partial/\partial x \wedge \partial/\partial y$. Let

---

[12] This is because, for our purposes, it will be enough to consider $N - 1$ or $N - 2$ monomials from $\Bbbk_k[\boldsymbol{x}]$ as arguments with the only remaining one or two arguments being polynomials: denote by and decompose the two polynomial arguments $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ into their constituent monomials. Now calculate the Wronskian of $N - 2$ monomial arguments and the remaining two arguments being a pair of constituent monomials from p and q, respectively. By Theorem 7 in this case, the same monomial can be produced with the opposite sign (thus it would cancel in the expansion by multilinearity) only if the two constituent monomials are swapped as arguments, as shown below.

Formally, decompose the two polynomials p, q into their constituent monomials: $p(\boldsymbol{x}) = c_1 a(\boldsymbol{x}) + c_2 b(\boldsymbol{x})$ and $q(\boldsymbol{x}) = d_1 a(\boldsymbol{x}) + d_2 b(\boldsymbol{x})$, where all four constants are non-zero. Expand the Wronskian of arguments p, q and $N - 2$ monomials from $\Bbbk_k[\boldsymbol{x}]$ by multilinearity. Denote by (a, b) the monomial obtained from the choice of arguments a, b and the same $N - 2$ monomials. Then the result of the expanded Wronskian is

$$c_1 d_1 (a, b) + c_2 d_1 (b, a) = (c_1 d_2 - c_2 d_1)(a, b).$$

If the expansion cancels, we obtain $c_1 d_2 = c_2 d_1$, from which it quickly follows that p and q are not linearly independent. As $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ are, by assumption, linearly independent, they must contain at least one pair of overall non-proportional monomial arguments. Then it suffices to consider as arguments of the Wronskian a pair of constituent monomials and $N - 2$ monomials of degree at most k, of which the Wronskian does not vanish identically after expansion; put these monomials as the new p and q themselves.

[13] More generally, for a polynomial $p(\boldsymbol{x})$ in place of $a(\boldsymbol{x})$, extend the results of the lemma by decomposing $p(\boldsymbol{x})$ into its constituent monomials and applying the lemma to each monomial.

$p(\boldsymbol{x}) = x^{n_1} y^{n_2}$ and $q(\boldsymbol{x}) = x^{m_1} y^{m_2}$ be arbitrary monomials. Then,

$$W^{k=1}_{d=2}(q, p, y) = \det \begin{vmatrix} 1 & 1 & 1 \\ m_1 & n_1 & 0 \\ m_2 & n_2 & 1 \end{vmatrix} \frac{pq}{x} = \{(n_2 - 1)m_1 - (m_2 - 1)n_1\} \frac{pq}{x}$$
$$= \{(n_1 + n_2 - 1)m_1 - (m_1 + m_2 - 1)n_1\} \frac{pq}{x},$$

where we were able to add n_1 and m_1 in the round brackets as they cancel by skew-symmetry of the coefficient, giving

$$= \{(\deg p - 1)m_1 - (\deg q - 1)n_1\} \frac{pq}{x}. \tag{47}$$

Notice how the coefficient intertwines the degrees of the two monomials p and q by the others' degree in x (i.e. $\deg(p)$ mixed with m_1), which is the replaced non-constant argument.

Example 14 ($k = 1$, any $d \geqslant 2$, $\widehat{x^j} \mapsto p$)**.** Now let the base dimension $d \geqslant 2$ be arbitrary and set the differential order to be 1; this gives the first order Wronskian $W^{k=1}_{d\geqslant 2} = \mathbf{1} \wedge \partial/\partial x^1 \wedge \cdots \wedge \partial/\partial x^d$. Replace 1 as argument by $q(\boldsymbol{x}) = (x^1)^{m_1} \cdots (x^d)^{m_d}$ and x^j by $p = (x^1)^{n_1} \cdots (x^d)^{n_d}$; by direct expansion (see Appendix A on p. 56) we obtain

$$W^{k=1}_{d\geqslant 2}(1, x^1, \ldots, x^{j-1}, p, x^{j+1}, \ldots, x^d) = \{(\deg p - 1)m_j - (\deg q - 1)n_j\} \frac{pq}{x^j}. \tag{48}$$

Notice (cf. Eq. 12 on p. 24 and the surrounding discussion) that if x^j does not divide the product pq, i.e. $n_1 = m_1 = 0$, then the coefficient vanishes identically (so the division by x^j is fictitious).

Example 15 ($d = 2$, $k = 2, 3, 4, 5$, $\widehat{x} \mapsto p$)**.** Fix the base dimension $d = 2$ with coordinates x, y and vary the total differential order $k \in \{2, 3, 4, 5\}$ (cf. the case $k = 1$ in Example 13). We now showcase computational results for the k values above:[14]

$$W^{k=2}_{d=2}(q, p, y, x^2, xy, y^2) = 2 \cdot (\deg p - 2)(\deg q - 2) \cdot \\ \cdot \{(\deg p - 1)m_1 - (\deg q - 1)n_1\} \frac{pq}{x} \tag{49a}$$

$$W^{k=3}_{d=2}(q, p, y, x^2, \ldots, xy^2, y^3) = 2^4\, 3 \cdot \prod\nolimits_{\ell=2,3} (\deg p - \ell)(\deg q - \ell) \cdot \\ \cdot \{(\deg p - 1)m_1 - (\deg q - 1)n_1\} \frac{pq}{x} \tag{49b}$$

$$W^{k=4}_{d=2}(q, p, y, x^2, \ldots, xy^3, y^4) = 2^{12}\, 3^4 \cdot \prod\nolimits_{\ell=2,3,4} (\deg p - \ell)(\deg q - \ell) \cdot \\ \cdot \{(\deg p - 1)m_1 - (\deg q - 1)n_1\} \frac{pq}{x} \tag{49c}$$

$$W^{k=5}_{d=2}(q, p, y, x^2, \ldots, xy^4, y^5) = 2^{26}\, 3^{10}\, 5 \cdot \prod\nolimits_{\ell=2,3,4,5} (\deg p - \ell)(\deg q - \ell) \cdot \\ \cdot \{(\deg p - 1)m_1 - (\deg q - 1)n_1\} \frac{pq}{x} \tag{49d}$$

[14] To compute the complete generalised Wronskian determinant symbolically, we write and run a script using the Python package SymPy; this Script 1 is given in Appendix C. The computations were performed on the home machine of the first author and on the computational cluster 'Hábrók' of the University of Groningen; see Appendix D for the raw computational outputs.

We emphasise that we have not normalised the standard monomials above (basically, not using $x^3y^2 \mapsto x^3y^2/3!\,2!$). Let us calculate the overall normalisation constant per degree:

$$\begin{array}{c|c|ll} \text{degree} & \text{monomials} & \text{product of normalisation constants} & \\ 2 & x, y & 1!\,1! & = 1 \\ 2 & x^2, xy, y^2 & 2!\,2! & = 2^2 \\ 3 & x^3, x^2y, xy^2, y^3 & 3!\,2!\,2!\,3! & = 2^4\,3^2 \\ 4 & x^4, x^3y, x^2y^2, xy^3, y^4 & 4!\,3!\,2!\,2!\,3!\,4! & = 2^{10}\,3^4 \\ 5 & x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5 & 5!\,4!\,3!\,2!\,2!\,3!\,4!\,5! & = 2^{16}\,3^6\,5^2, \end{array} \tag{50}$$

Observe that dividing the product of normalisation degree from degree 1 up to k by $k!\,(k-1)!$ the overall coefficients in Eqs (49) are obtained (for instance, $2^{2+4+10}\,3^{2+4}/4!\,3! = 2^{12}\,3^4$ for $k=4$ and $2^{2+4+10+16}\,3^{2+4+6}\,5^2/5!\,4! = 2^{26}\,3^{10}\,5$ for $k=5$).

This results for $k \in \{2,3,4,5\}$ into the formula:

$$W_{d=2}^{k\in\{2,3,4,5\}}\Big(q, p, \frac{y}{1!}, \frac{x^2}{2!}, \frac{xy}{1!\,1!}, \ldots, \frac{y^k}{k!}\Big) = \frac{1}{k!\,(k-1)!}\prod_{\ell=2}^{k}(\deg p - \ell)(\deg q - \ell)\cdot$$
$$\cdot\{(\deg p - 1)m_1 - (\deg q - 1)n_1\}\frac{pq}{x}, \tag{51}$$

which – we show in Theorem 16 – is indeed the general form for any $k \geqslant 1$. (The empty product $\prod_{\ell=2}^{1}$ is interpreted as the unit: $\prod_{\varnothing} = 1$.)

Theorem 16 (Main formula)**.** *Take any dimension $d \geqslant 1$ and order $k \geqslant 1$; set $N = \binom{d+k}{k}$. Let $p(\boldsymbol{x}), q(\boldsymbol{x}) \in \Bbbk_{k+1}[\boldsymbol{x}]$ be arbitrary monomials: $p(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{n}}} = (x^1)^{n_1}\cdots(x^d)^{n_d}$ and $q = \boldsymbol{x}^{\vec{\boldsymbol{m}}} = (x^1)^{m_1}\cdots(x^d)^{m_d}$. Calculating the complete generalised Wronskian (of differential order k over dimension d) of all monomials up to degree k as arguments, but with the constant polynomial $1 = \boldsymbol{x}^{\vec{\boldsymbol{r}}_1}$ replaced by the monomial $q(\boldsymbol{x})$ and $x^1 = \boldsymbol{x}^{\vec{\boldsymbol{r}}_2}$ by $p(\boldsymbol{x})$, gives*

$$W_d^k\Big(\underbrace{q(\boldsymbol{x})}_{\widehat{1}}, \underbrace{p(\boldsymbol{x})}_{\widehat{x^1}}, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, \ldots, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!}\Big) = \frac{1}{k!(k-1)!}\prod_{\ell=2}^{k}(\deg p - \ell)(\deg q - \ell)\cdot$$
$$\cdot\{(\deg p - 1)m_1 - (\deg q - 1)n_1\}\cdot\frac{pq}{x^1}, \tag{52}$$

where, of course, if $x^1 \nmid pq$, then the coefficient is zero by virtue of $m_1 = n_1 = 0$.[15]

Problem 2 (vanishing mechanism in the Main formula (16))**.** Let $p(\boldsymbol{x}) = (x^1)^{n_1}\cdots(x^d)^{n_d}$ and $q(\boldsymbol{x}) = (x^1)^{m_1}\cdots(x^d)^{m_d}$ be arbitrary monomials over base dimension $d \geqslant 2$. Let the degree of the generalised Vandermonde determinant be $k \geqslant 1$. Consider all monomials up to degree k: $1, x^1, \ldots, x^d, (x^1)^2, x^1x^2, \ldots, (x^d)^k$; let their power d-tuples be $\vec{\boldsymbol{r}}_1 = (0,\ldots,0)$, $\vec{\boldsymbol{r}}_2 = (1,0,\ldots 0)$, $\vec{\boldsymbol{r}}_3 = (0,1,0,\ldots,0)$, ..., $\vec{\boldsymbol{r}}_N = (0,\ldots,0,k)$, and let $\vec{\boldsymbol{n}} = (n_1,\ldots,n_d)$ and $\vec{\boldsymbol{m}} = (m_1,\ldots,m_d)$ be the power d-tuples of p and q, respectively.

Calculate their Wronskian as in Theorem 16; by formula (52) it vanishes. *What is the (algebraic) mechanism*[16] *for the complete generalised Vandermonde determinant of the power d-tuples,* $\det \mathrm{Van}_d^k(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{r}_3, \ldots, \boldsymbol{r}_N) = 0$, *to vanish* in each of the following cases:

[15] Cf. Footnote 20 on p. 48 for a simplification of notation of the curly bracket part of the main formula (52).

[16] Namely, are some (specific) rows or columns (clearly) linearly dependent as a result, or which terms cancel during the expansion?

(*i*) if the degrees in x^1 of $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ are zero, $n_1 = m_1 = 0$ (i.e. $x^1 \nmid pq$);
(*ii*) if the total degrees and the degrees in x^1 of $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ are equal, $\deg p = \deg q$ and $\deg_1 p = \deg_q q$?

Refer to Theorem 7 on p. 7 about the complete generalised Vandermonde determinant appearing as the coefficient. For case *(i)* refer to the deficiency condition *(iii)* from Proposition 9 on p. 32 for the Vandermonde determinant to vanish.

Remark 13. Direct expansion of the complete generalised Wronskian determinant in main formula 52, or, effectively, of the complete generalised Vandermonde determinant, is very difficult over base dimension $d \geqslant 2$ if the differential order is $k \geqslant 2$. The proof of Theorem 16 by direct expansion in the case $k = 1$ is given in Appendix A.

Proof (of Theorem 16). Recall that by Theorem 7 the result of the bracket is the required monomial, whose coefficient is the complete generalised Vandermonde determinant. The proof strategy is to view this Vandermonde determinant as a polynomial in the degrees of monomials p and q. A polynomial of degree (at most) k can have (if they exist, i.e. all roots are in the base field) at most k linear factors in the base variables. In this proof we find *all* factors of the polynomial, hence we decompose the polynomial as a product of the linear factors. Of the k factors, $k - 1$ factors will be immediate and do not mix the degrees of the monomials p and q, whereas the final kth factor (see the curly bracket in formula (52)) mixes the degrees and will require significant work to find. We then finally find the universal (independent of p and q) constant in front of the linear factors.

The case $k = 1$ is proven in Appendix A. Let $k \geq 2$, albeit the proof still holds for $k = 1$. Calculate the resulting monomial by 7: the pq/x term is obtained by adding the monomials' degrees, for which the coefficient is the generalised complete Vandermonde determinant. We view the result of the determinant as a polynomial in $\boldsymbol{m} = (m_1, ..., m_d)$; expansion over the first column yields a polynomial of (total) degree k. Therefore, this polynomial in $\boldsymbol{m}$ has at most k linear factors (in $\boldsymbol{m}$).

If the column of $\boldsymbol{m}$'s is equal to another column, the determinant must be zero; this introduces a linear factor. As the set of columns of degrees at least two is complete, equality to another column of degree at least 2 is equivalent to $2 \leq \deg(q) \leq k$; this introduces $k-1$ linear factors of the form $(\deg q - \ell)$ for $\ell = 2, ..., k$ hence their (l.c.m.) product $\prod_{\ell=2}^{k}(\deg(q) - \ell)$.

Instead viewing the determinant as a polynomial in variables $\boldsymbol{n} = (n_1, ..., n_d)$ introduces, by the mechanism above applied to p, $k - 1$ linear factors in $\boldsymbol{n}$'s of the form $(\deg p - \ell)$ for $\ell \in \{2, ..., k\}$. The factors involving the degrees in q and the degrees in p do not mix, hence we their product, $\prod_{\ell=2}^{k}(\deg p - \ell) \prod_{\ell=2}^{k}(\deg q - \ell)$, is a factor of the determinant.

Viewing the determinant as a polynomial either in $\boldsymbol{m}$'s or $\boldsymbol{n}$'s, it follows that only one other linear factor in $\boldsymbol{m}$'s (or, equivalently, $\boldsymbol{n}$'s) is possible; the factor may mix the individual components of $\boldsymbol{m}$ and $\boldsymbol{n}$. *We now find this factor;* denote it by the function $f(\boldsymbol{n}, \boldsymbol{m})$:

$$W_d^k\Big(q, p, \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_2}}{\vec{\boldsymbol{r}}_2!}, ..., \frac{\boldsymbol{x}^{\vec{\boldsymbol{r}}_N}}{\vec{\boldsymbol{r}}_N!}\Big) = (\text{const}) \prod_{\ell=2}^{k}(\deg p - \ell)(\deg q - \ell) \cdot f(\boldsymbol{n}, \boldsymbol{m}) \cdot \frac{pq}{x^1}, \tag{53}$$

where the constant[17] (const) is independent of $\boldsymbol{n}$ and $\boldsymbol{m}$. The function $f(\boldsymbol{n},\boldsymbol{m})$ has degree one in $\boldsymbol{n}$'s and also degree one in $\boldsymbol{m}$'s.

The factor $f(\boldsymbol{n},\boldsymbol{m})$ cannot equal a previously factor (i.e. of the form $(\deg(\cdot)-\ell)$) as the choice of monomials $p=q$, both of degree $k+1$, yields zero in (53), which is not captured by the previous factors, which do not mix $\boldsymbol{n}$'s and $\boldsymbol{m}$'s; hence this root must be captured by the factor $f(\boldsymbol{n},\boldsymbol{m})$. The factor $f(\boldsymbol{n},\boldsymbol{m})$ of degree-one in each of the two d-tuples of arguments mixes $\boldsymbol{m}$'s and $\boldsymbol{n}$'s, hence written in full generality, is

$$f(\boldsymbol{n},\boldsymbol{m}) = \sum_{i=1}^{d} \lambda^{i,0} n_i + \sum_{j=1}^{d} \lambda^{0,j} m_j + \sum_{i,j} \lambda^{ij} n_i m_j + C,$$

with all λ factors and C constants. Notation: $\boldsymbol{n} = (n_1, ..., n_d)$ and $\boldsymbol{m} = (m_1, ..., m_d)$. For convenience, the indices i and j (typically, $i<j$) will typically range from 1 to d, whereas the indices μ and ν will incorporate also the 0 value: $\mu,\nu \in \{0,1,...,d\}$.

As the determinant is anti-symmetric upon an exchange of p and q, or, equivalently, the swap $n_i \rightleftarrows m_i$ for all $i \in \{1,...,d\}$, but all factors of $(\deg(\cdot)-\ell)$ are symmetric in $n_i \leftrightarrows m_i$, it follows that f is skew w.r.t the two d-tuple arguments. Therefore $\lambda^{\mu\nu} = -\lambda^{\nu\mu}$ and $C=0$ for all indices $\mu,\nu \geq 0$; hence diagonal terms vanish: $\lambda^{ii}=0$. The function f thus becomes

$$f(\boldsymbol{n},\boldsymbol{m}) = \sum_{i=1}^{d} \lambda^{i,0}(n_i - m_i) + \sum_{i<j} \lambda^{ij}(n_i m_j - n_j m_i).$$

The choices of d-tuples $\boldsymbol{n} = (0,...,0)$ and $\boldsymbol{m} = (0,1,0,...,0)$ corresponding to the monomials $p=1$ and $q=y$, which in formula (53) give zero as the argument y is repeated. But none of the terms under the product in (53) are zero, hence the vanishing condition must come from a root of the term $f(\boldsymbol{n},\boldsymbol{m})$:

$$0 = f(\boldsymbol{n}=\boldsymbol{0}, m_2=1) = \textstyle\sum_{i=1}^{d}(-)\lambda^{i,0} m_i = -\lambda^{2,0}\cdot 1,$$

which implies $\lambda^{2,0}=0$; similarly, all other zero-index constants vanish: $\lambda^{\mu,0}=0$ for indices $\mu \geq 2$ by choosing the d-tuple $\boldsymbol{m}$ to be $m_\mu \neq 0$ and $m_\nu = 0$ otherwise ($\nu \neq \mu$). This yields

$$f(\boldsymbol{n},\boldsymbol{m}) = -\lambda^{0,1}(n_1 - m_1) + \sum_{i<j} \lambda^{ij}(n_i m_j - n_j m_i) \tag{54}$$

after applying $\lambda^{1,0} = -\lambda^{0,1}$.

Let us now put the monomials $q=x$ and $p=y$ as arguments (hence $\boldsymbol{m}=(1,0,...,0)$ and $\boldsymbol{n}=(0,1,0,...,0)$); this choice yields a vanishing r.-h.s. in (53) as y is a repeated argument. But again none of the factors in the products vanish, thus the vanishing condition is due to:

$$0 = f(n_2=1, m_1=1) = -\lambda^{0,1}(-)m_1 + \lambda^{1,2}(-)n_2 m_1 = \lambda^{0,1} - \lambda^{1,2},$$

which implies $\lambda^{1,2} = \lambda^{0,1}$; similarly, we obtain $\lambda^{1,\mu} = \lambda^{0,1}$ for every index $\mu \in \{2,...,d\}$ by putting as argument the monomial $p = x^\mu$. Split the sum in the term (54) into two sums

[17] This constant is not identically zero as there exists a choice of monomials, $q(\boldsymbol{x})=1$ and $p(\boldsymbol{x})=x$, which give a non-zero r.-h.s. in (53), namely, the constant unit polynomial.

according to whether the summation variable i is equal to one, that is, $i = 1$ and $1 < i < j$:

$$\sum_{i<j} \lambda^{ij}(n_i m_j - n_j m_i) = \sum_{j=2}^{d} \lambda^{1,j}(n_1 m_j - n_j m_1) + \sum_{1<i<j} \lambda^{ij}(n_i m_j - n_j m_i),$$

which gives, after applying the substitution $\lambda^{1,j} = \lambda^{0,1}$ found above, the expression

$$\begin{aligned} f(\boldsymbol{n},\boldsymbol{m}) &= \lambda^{0,1}\Big\{-(n_1 - m_1) + \sum_{j=1}^{d}(n_1 m_j - n_j m_1)\Big\} + \sum_{1<i<j} \lambda^{ij}(n_i m_j - n_j m_i) \\ &= \lambda^{0,1}\{(\deg p - 1)m_1 - (\deg q - 1)n_1\} + \sum_{1<i<j} \lambda^{ij}(n_i m_j - n_j m_i), \end{aligned} \tag{55}$$

where we used that $\deg(p) = n_1 + \cdots + n_d$ and $\deg(q) = m_1 + \cdots + m_d$. Notice that the first summand is exactly the curly bracket in the main formula (52) up to an overall constant $\lambda^{0,1}$. We now show the sum comprising the second summand vanishes.

Put as arguments in Eq. (52) the monomials $p(\boldsymbol{x}) = y^{k+1}$ and $q(\boldsymbol{x}) = y^{k+2}$; they are not divisible by the $x = x^1$ coordinate, namely, $n_1 = 0 = m_1$; hence the pq/x term is not a polynomial, and condition *(iii)* from Proposition 9 on p. 32 implies that the coefficient in front of pq/x vanishes. But as $\deg p, \deg q > k$ none of the products in formula (53) are zero; this yields another root of the factor $f(\boldsymbol{n},\boldsymbol{m})$:

$$0 = f(\boldsymbol{n},\boldsymbol{m}) = \sum_{i<i<j} \lambda^{ij}(n_i m_j - n_j m_i),$$

where the the curly bracket in (55) vanished by $n_1 = 0 = m_1$. We therefore obtain

$$f(\boldsymbol{n},\boldsymbol{m}) = \lambda^{0,1}\{(\deg p - 1)m_1 - (\deg q - 1)n_1\},$$

where $\lambda^{0,1}$ is a universal constant independent of $\boldsymbol{n}$ and $\boldsymbol{m}$. We now proceed to find it.

Putting the arguments p and q to be the monomials they replaced, that is, $q = 1$ and $p = x$, we know that the result of the Wronskian in Eq. (52) must yield unit (as product of units on the diagonal), hence obtaining overall that

$$1 = (\text{const}) \cdot \prod_{\ell=2}^{k}(1-\ell) \prod_{\ell=2}^{k}(-\ell) \cdot \{0+1\},$$

which gives $(\text{const}) = 1/k!\,(k-1)!$. We thus recover the formula in (52), which completes the proof. □

Remark 14. The reason ℓ starts in formula (52) from $\ell = 2$ is that the first degree for which the arguments in (52) contain *all* the monomials of that degree[18] is 2. This is also why in formula (37) on p. 37 the second product runs from $\ell = r_j + 1$ to $\ell = k$: the set of monomials of degree r_j lacks the monomial $\boldsymbol{x}^{\vec{\boldsymbol{r}}_j}/\vec{\boldsymbol{r}}_j!$, whereas all higher degrees are complete.

Let us give a few computational examples illustrating this delayed start of ℓ at the first complete degree in the set of monomial arguments.

[18]This is equivalently referred to as a *complete* degree; refer to the Definition 3 on p. 23 of the *complete* generalised Wronskian, which is complete in all degrees $0, 1, \ldots, k$ by the monomial–differential-operator correspondence (basically, $\partial^3/\partial x^2 \partial y \leftrightarrows x^2 y$).

Example 16 ($d = 2$, $k = 2,3,4,5$, $\widehat{(x)^2} \mapsto p$)**.** Fix the base dimension $d = 2$ and vary the total differential order $k \in \{2,3,4,5\}$ of the complete generalised Wronskian W_d^k as in Example 15 on p. 43; adopt the notation $p(\boldsymbol{x}) = x^{n_1}y^{n_2}$ and $q(\boldsymbol{x}) = x^{m_1}y^{m_2} \in \Bbbk[\boldsymbol{x}]$. As arguments, take the standard monomials $1, x, y, x^2, xy, \ldots, xy^{k-1}, y^k$, but replace 1 by the monomial $q(\boldsymbol{x})$ and $(x)^2 = xx$ by $p(\boldsymbol{x})$.[19] Put, by definition,[20]

$$\{p,q\}_{\widehat{xx}} := \{(\deg p-1)(\deg p-2)\, m_1(m_1-1) - (\deg q-1)(\deg q-2)\, n_1(n_1-1)\}\frac{pq}{xx}, \quad (57)$$

which we use to simplify the formulae below. Computations using Script 1 in Appendix C, setting the optional argument `prepl=(x,x,)`, gave:[21]

$$W_{d=2}^{k=2}(q,x,y,p,xy,y^2) = -1\cdot\{p,q\}_{\widehat{xx}}, \quad (58a)$$

$$W_{d=2}^{k=3}(q,x,y,p,xy,\ldots,xy^2,y^3) = -2^4\,3\cdot(\deg p-3)(\deg q-3)\cdot\{p,q\}_{\widehat{xx}} \quad (58b)$$

$$W_{d=2}^{k=4}(q,x,y,p,xy,\ldots,xy^3,y^4) = -2^{11}\,3^5\prod_{\ell=3,4}(\deg p-\ell)(\deg q-\ell)\cdot\{p,q\}_{\widehat{xx}}, \quad (58c)$$

$$W_{d=2}^{k=5}(q,x,y,p,xy,\ldots,xy^4,y^5) = -2^{27}\,3^{10}\,5\prod_{\ell=3,4,5}(\deg p-\ell)(\deg q-\ell)\cdot\{p,q\}_{\widehat{xx}} \quad (58d)$$

Notice how the factor in the curly brackets $\{p,q\}_{\widehat{xx}}$ is the same in all equations (58), whereas the product ranges from $\ell = 3$ to $\ell = k$; the same mechanism as explained in Remark 14.

We contrast the coefficient factor (which mixes the degrees of p and q) in the curly brackets in formulas (58) above against formulas (49) in Example 15 on p. 43:

$$\{p,q\}_{\widehat{x}} := \{(\deg p-1)m_1 - (\deg q-1)n_1\}\frac{pq}{x}, \qquad \text{(this is from formulas (49))}$$

$$\{p,q\}_{\widehat{xx}} := \{(\deg p-1)(\deg p-2)\, m_1(m_1-1) - (\deg q-1)(\deg q-2)\, n_1(n_1-1)\}\frac{pq}{xx}.$$

It is evident that the replacement $\widehat{x} \mapsto p$ versus $\widehat{xx} \mapsto p$ turns one linear factor in the degrees n_1, n_2 of p into $(\deg p-2)$ and (n_1-1) inside the curly bracket; recall that the sum of the factors' total degrees in n_1 and n_2 must be k (see commentary within the proofs of Theorems 13 and 16).

This suggests that replacing a higher-degree term by $p(\boldsymbol{x})$ (for instance, $\widehat{(x)^3}, \widehat{(x)^4}, \ldots, \widehat{(x)^k}$, for which one can easily *conjecture* the factor in the equivalent curly brackets,

$$\{p,q\}_{\widehat{(x)^r}} \stackrel{?}{=} \left\{\prod_{\ell=1}^{r}(\deg p-\ell)\prod_{\ell=0}^{r-1}(m_i-\ell) - \prod_{\ell=1}^{r}(\deg q-\ell)\prod_{\ell=0}^{r-1}(n_i-\ell)\right\}\frac{pq}{(x)^r}, \quad (59)$$

or $\widehat{xy}$, $\widehat{x^2y}$, $\widehat{x^2y^2}$ and so on[22]) will result in fewer terms of the form $(\deg p-\ell)$, and the terms within the curly bracket will be of more complicated form; see Problem 4 and the accompanying Commentary.

---

[19]In contrast, p replaced x in Example 15.

[20]In contrast, in Example 15 and the main formula (52) on pp. 43 and 44, resp., to simplify the notation we could have put

$$\{p,q\}_{\widehat{x}} := \{(\deg p-1)m_1 - (\deg q-1)n_1\}\frac{pq}{x}. \quad (56)$$

There at least, unlike here, the formulae were manageable.

[21]The raw outputs are given in Appendix D.

[22]We do not yet know what will be there.

Open problem 3. The structure constants for our finite-dimensional N-ary Lie algebras can be written as a differential operator acting on the $(k+1)$th degree argument – see Eq. (38) on p. 37. Can the skew-symmetric operators in the curly brackets – Eqs. (56), (57), and (59) – be written as differential operators acting on the (poly- or) monomials p and q?

Remark 15 (Vandermonde factorisations take the simplest form). We have so far explained the nature of every factor in formula (37) from Theorem 13 and in formula (52) in Theorem 16; this is the fullest and simplest possible factorisation of the generalised Vandermonde determinants which we can provide (cf. formula (14) in Theorem–Definition 2 when the base dimension is $d = 1$). Now, for $d \geqslant 2$, all the factors which we actually present have appeared either due to:

(i) the monomial p coinciding with a monomial of degree ℓ from the range $\{r+1, \ldots, k\}$, where r is the degree of the monomial, which p replaced as argument;
(ii) the monomial p coinciding with a monomial of degree less or equal to r;
(iii) (specific only to Eq. (52)) the two monomials p and q are interrelated in such a way that the curly bracket in (52) vanishes.

If three or more of the arguments were arbitrary monomials, we should expect a totally-skew factor mixing the degrees of the three arbitrary monomials. It is an open problem.

Open problem 4 (factorisations of the generalised Vandermonde determinant)**.** Put the base dimension $d \geqslant 2$. Suppose that the number of standard monomials (from the set 1, x^1, ..., x^d, $(x^1)^2$, x^1x^2, ..., $(x^d)^k$) replaced by the new (arbitrary) monomials $p(\boldsymbol{x})$ and $q(\boldsymbol{x})$ is increased to three or more *monomials* (i.e. $p_1 = p$, $p_2 = q$, p_3, p_4 and so on). What is (or are) the simplest factorisation(s) of the complete generalised Vandermonde determinant of the arguments' power d-tuples (relative to the complete generalised Wronskian $W^{k\geqslant 1}_{d\geqslant 1}$)?

Note that the answer depends on the choice of standard monomials to be replaced by the new ones.

Comment (to Open problem 4)**.** In this paper, specifically Theorems 13 and 16, we explored the cases when:

(i) any single monomial from the standard set is replaced;
(ii) two monomials are replaced, specifically 1 and x^1 (equivalently by symmetry, 1 and x^i for any $i \in \{1, \ldots, d\}$).

Replacing other pairs of monomials from the standard set of degrees not exceeding $k \geqslant 1$ over base dimension $d \geqslant 1$ will result in other formulas and other factorisations of the complete generalised Vandermonde determinants: see Example 16.

F. Brown in [1] expresses the generalised – **NB!** 'generalised' in a way still more general than ours![23] – Vandermonde determinant (of arbitrary monomial arguments) as sums of smaller-size generalised Vandermonde determinants. In the sums the terms do not visibly share any common factor.

We now state our second main theorem. For relevant definitions and details see Definitions 8 and accompanying Lemma 11 on pp. 34 and 34, resp., about the different types of N-ary polynomial SH-Lie algebras, and why every *consistent* (cf. 10) algebra is either trivial,

[23]Our *complete* generalised Vandermonde determinants correspond to those in §2.4 of [1]. The further generalisation drops the 'complete' from the complete generalised Vandermonde determinant; the Vandermonde determinant is now with respect to an *in*complete Wronskian (cf. [8] for definitions and a discussion).

lonely, chubby, or lanky; see also Footnote 12 on p. 42 for the reasoning why it is sufficient to consider algebras spanned by pure monomials. In the first main Theorem 14 we showed that every lonely algebra is finite-dimensional of dimension $N+1$; trivial algebras are trivially of dimension at most N. In the proof of the theorem we extensively use the main formula (52).

Theorem 17 (main theorem-**2**: *no other* consistent finite-dimensional algebras)**.** *Take any base dimension $d \geqslant 1$ and differential order $k \geqslant 1$; set $N = \binom{d+k}{k}$. Put the* <u>complete</u> *generalised Wronskian determinant W_d^k as the N-ary bracket. Suppose that an N-ary polynomial SH-Lie algebra $\mathcal{A}$ is* consistent, *but* <u>not lonely</u>*; i.e. $\mathcal{A}$ is either of the two options:*

(i) $\mathcal{A}$ is chubby, that is, $\mathcal{A} \supseteq \Bbbk_k[\boldsymbol{x}] \oplus \langle p(\boldsymbol{x}), q(\boldsymbol{x})\rangle$, where $p \not\propto q$ are (poly- or) monomials of degrees $\deg(p) = k+1 = \deg(q)$;

(ii) $\mathcal{A}$ is lanky, that is, $\mathcal{A}$ contains the monomial $(x^i)^{k+\ell}$, where $\ell \geqslant 2$, and thus the tower of monomials in a single coordinate x^i: $\mathcal{A} \supseteq \Bbbk_k[\boldsymbol{x}] \oplus \langle (x^i)^{k+1}, \ldots, (x^i)^{k+\ell}\rangle$.

Then $\mathcal{A}$ <u>cannot</u> *be finite-dimensional, so $\dim_{\Bbbk}(\mathcal{A}) = +\infty$.*

Proof. We show that every (*i*) chubby, (*ii*) lanky algebra is infinite-dimensional by producing a sequence of monomials, constructed using the Wronskian determinant, which diverge in degree going to infinity. That is, the algebra contains this infinite sequence of independent monomials, hence is infinite dimensional. All cases proceed via induction.

• **Case: chubby** (*i*)**.** Let $p \not\propto q$ be monomials of degree $k+1$.; put $p(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{n}}} = (x^1)^{n_1} \cdots (x^d)^{n_d}$ and $q(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{m}}} = (x^1)^{m_1} \cdots (x^d)^{m_d}$. As the monomials are not proportional to each other, there must exist a coordinate, in which their powers differ: $n_i \neq m_i$. Without loss, put $i = 1$ (if needed, by rearrangement of the order of coordinates). We now separate the possible cases of n_1 and m_1: firstly, $m_1 > 1$ splits into $n_1 = 0$, $m_1 > n_1 \geqslant 1$, or $n_1 > m_1$; in the last case, swap the monomials p and q, such that m_1 is always bigger than n_1. Secondly, if $m_1 = 1$, we have $n_1 = 0$ or $n_1 > 1$; again, in the last case swap $p \leftrightarrows q$, obtaining $m_1 > n_1$. Finally, if $m_1 = 0$, then $n_1 > 0$; swapping $p \leftrightarrows q$ we again obtain a previous case. Hence, without loss, an exhaustive list of cases is:

(*a*) $m_1 > 1$ and $n_1 = 0$; (*b*) $m_1 > n_1 \geqslant 1$; (*c*) $m_1 = 1$ and $n_1 = 0$.

We tackle each separately. Denote $x := x^1$. In all cases put $p_0 = p$, $p_1 = pq/x$, $p_2 = p_1 q/x$, ..., $p_n = p_{n-1}q/x$, ...; note that the division does produce a monomial as $x|q$ in all cases ($m_1 \geqslant 1$). As the total degree $\deg(p) = k+1$ and $\deg(p_1) = 2(k+1) - 1 = 2k+1$, we inductively find $\deg(p_n) = (n+1)k+1$. Indeed, as $n \to +\infty$, it follows that $\deg(p_n) \to +\infty$. Our goal is to produce each p_n by using arguments from $\mathcal{A}$ and previous $p_0, p_1, \ldots, p_{n-1}$.

Firstly, note that whenever the degrees of the two free arguments in the Wronskian as in the main formula (52) are $\deg(p), \deg(q) > k$, the products of $(\deg(\cdot) - \ell)$ over $\ell \in \{2, 3, \ldots, k\}$ do not vanish. Put $c := 1/[k!\,(k-1)!] \cdot \prod_{\ell=2}^{k}(\deg(p) - \ell)(\deg(q) - \ell) \neq 0$ for every choice of p and q of degrees $\deg(p), \deg(q) > k$.

Subcase (*a*)**.** In this case $\deg_x(q) = m_1$ and, inductively, $\deg_x(p_n) = n(m_1 - 1)$. Then

$$W_d^k\Big(q, p, x^2, \ldots, \frac{(x^d)^k}{k!}\Big) = c \cdot \{(k+1-1)m_1 - 0\}\frac{pq}{x} = c \cdot k m_1 \frac{pq}{x},$$

where we note that the coefficient does not vanish, and

$$W_d^k\Big(q, p_n, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c \cdot \{((n+1)k+1-1)m_1 - (k+1-1)\,n(m_1-1)\}\frac{p_n q}{x} = c \cdot k(m_1+n)\frac{p_n q}{x},$$

where again the coefficient does not vanish. By induction that we thus obtain that each monomial p_n belongs to the algebra $\langle p_0, p_1, p_2, ..., p_n, ...\rangle \subseteq \mathcal{A}$. As all p_n's are linearly independent, we find that $\dim_\Bbbk \mathcal{A} = +\infty$.

Subcase (*b*). Now $\deg_x(pq/x) = m_1 + n_1 - 1 \geqslant 2$ and, inductively, we see that the degree in $x = x^1$ of the nth monomial $p_n(\boldsymbol{x})$ is $\deg_x(p_n) = n(m_1-1)+n_1$. Then

$$W_d^k\Big(q, p, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c \cdot \{(k+1-1)m_1 - (k+1-1)n_1\}\frac{pq}{x} = c \cdot (m_1-n_1)\frac{pq}{x},$$

where the coefficient is not zero, and

$$W_d^k\Big(q, p_n, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c \cdot \{((n+1)k+1-1)m_1 - (k+1-1)(n(m_1-1)+n_1)\}\frac{p_n q}{x} = c \cdot k(m_1-n_1+n)\frac{p_n q}{x},$$

where again the coefficient does not vanish. We see that $\dim_\Bbbk \mathcal{A} = +\infty$.

Subcase (*c*). In this case $\deg_x(p_n) = 0$. Then

$$W_d^k\Big(q, p, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c \cdot \{(k+1-1)m_1 - 0\}\frac{pq}{x} = c \cdot km_1\frac{pq}{x},$$

where the coefficient does not vanish, and

$$W_d^k\Big(q, p_n, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c \cdot \{((n+1)k+1-1)m_1 - 0\}\frac{p_n q}{x} = c \cdot (n+1)k \cdot \frac{p_n q}{x},$$

where the coefficient is never zero. Again thus $\dim_\Bbbk \mathcal{A} = +\infty$.

• **Case: lanky** (*ii*). Put, without loss, $q(\boldsymbol{x}) = (x^1)^{k+1}$ and $p(\boldsymbol{x}) = (x^1)^{k+\ell}$, where the degree $\ell \geqslant 2$. Then $\deg(pq/x) = k+1+k+\ell-1 = 2k+\ell$ and, inductively, as the monomials involve only one coordinate: $\deg p_n = \deg_x p_n = (n+1)k+\ell$. Then

$$W_d^k\Big(q, p, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c\{(k+\ell-1)(k+1) - (k+1-1)(k+\ell)\}\frac{pq}{x} = c(\ell-1)\frac{pq}{x},$$

where $\ell - 1 > 0$, hence the coefficient does not vanish; and

$$W_d^k\Big(q, p_n, x^2, ..., \frac{(x^d)^k}{k!}\Big) = c\{((n+1)k+\ell-1)(k+1) - (k+1-1)((n+1)k+\ell)\}\frac{p_n q}{x} = c(nk+\ell-1)\frac{p_n q}{x},$$

where the coefficient is never zero. We have found that $\langle p_0, p_1, ..., p_n, ...\rangle \subseteq \mathcal{A}$, so the dimension of the polynomial SH-Lie algebra $\mathcal{A}$ is $\dim_\Bbbk \mathcal{A} = +\infty$.

All possible cases have been exhausted, whereby in each case $\dim_\Bbbk \mathcal{A} = +\infty$. The proof is thus complete. $\square$

Remark 16. From the proof of the second main Theorem 17 and Theorems 7 and 16 we see that to keep the algebra $\mathcal{A}$ finite dimensional, we must have a bound on the polynomial degrees (and their sum) of all the arguments in the bracket. To describe that bound, we recall that the sum of degrees minus the shift (see Definition 7 and Corollary 8 on p. 32) cannot exceed the highest degree of a monomial in the set. And if the bound is exceeded, the only way for the algebra $\mathcal{A}$ to remain finite-dimensional is the vanishing of the coefficient of the resulting monomial.

Our computational experiments (some 20–30 cases done by the first author) show that accidentally hitting the zero coefficient is very unlikely (unless it is an instance of repeated arguments or an instance of deficiency: the shift down of the degree is greater than the sum of available degrees; see Proposition 9 (*iii*)). Moreover, we observe that three always is a set of arguments, reached through iterated commutation, such that the coefficient is not zero and the resulting degree exceeds that of any argument. (In fact, most choices work.)

Open problem 5 (*in*-consistent finite-dimensional algebras)**.** Relax the consistency assumption in the main Theorems 14 and 17: suppose that the set of generators of the N-ary polynomial SH-Lie algebra $\mathcal{A}$ has a gap somewhere in polynomial degrees not exceeding k. But still let us require that the algebra is *perfect*: $\operatorname{span}_{\Bbbk} W_d^k(\mathcal{A},\ldots,\mathcal{A}) = \mathcal{A}$.[24]

Can $\mathcal{A}$ be finite-dimensional? (So far, no such example is known; the task is to either construct one for some choice of values d, k, or prove that no such example exists.)

Conjecture 18 (to Open problem 5: 'no')**.** We expect that the answer to the question in Open problem 5 is *no*: over dimension $d \geqslant 1$, *in-consistent* and *perfect* polynomial strongly homotopy Lie algebras with complete generalised Wronskian brackets may only be infinite-dimensional.

Conclusion.

We have found *all finite-dimensional* N-ary *consistent* polynomial SH-Lie algebras $\mathcal{A}$: that is, $\Bbbk_k[x^1,\ldots,x^d] \subseteq \mathcal{A} \subsetneq \Bbbk[x^1,\ldots,x^d]$ with the *complete* generalised Wronskian determinant – of differential order k over base dimension d – as the N-ary bracket, $N = \binom{d+k}{k}$. Each such algebra has dimension $(\leqslant)N$ or $N+1$: trivial, $\mathcal{A} \subseteq \Bbbk_k[\boldsymbol{x}]$, or lonely, $\mathcal{A} = \Bbbk_k[\boldsymbol{x}] \oplus \langle p(\boldsymbol{x})\rangle$, algebras, respectively, where $p(\boldsymbol{x}) \in \Bbbk_{k+1}[\boldsymbol{x}]$ is any homogeneous polynomial of degree $k+1$.

Discussion.

We leave to future research the investigation of very general properties of N-ary SH-Lie polynomial algebras. Fix the base dimension $d \geqslant 1$ and the differential order $k \geqslant 1$ of the complete generalised Wronskian W_d^k; set $N = \binom{d+k}{k}$. Consider an SH-Lie polynomial algebra $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$ with W_d^k as the N-ary bracket.

Let us make some preliminary steps to phrase Definition 9 below. Write each polynomial in $\mathcal{A}$ as a linear combination of monomials. Firstly, suppose all such monomials do belong to the algebra. Define the *maximal degree sum* in a coordinate x^i by N linearly independent

[24] By requesting that $\mathcal{A}$ is perfect we exclude trivial cases such as $\mathcal{A} = \operatorname{span}_{\Bbbk}\langle 1, y, y^2, \ldots, y^{100}\rangle$ over base dimension $d = 2$, $\mathbb{R}^2 \ni (x,y)$, and differential order $k = 10$: all Wronskian brackets by $W_{d=2}^{k=10}$ are trivially zero, so the algebra $\mathcal{A}$, generated by these powers of y, is closed and finite-dimensional. We also exclude the trivial cases when $\dim(\mathcal{A}) = N$ and the result of the bracket of all N generators again belongs to the algebra $\mathcal{A}$.

monomial arguments, $D_i^{(N)}(\mathcal{A})$, to be:

$$D_i^{(N)}(\mathcal{A}) = \max_{\substack{a_1,\ldots,a_N \in \mathcal{A} \\ \text{lin. indep. monoms}}} \sum_{j=1}^{N} \deg_i(a_j). \tag{60}$$

(See Definition 7 on p. 31 for the definition of degree in x^i.)

We extend the definition of the maximal degree sum to the case where polynomials in $\mathcal{A}$ are not always decomposed into monomials again belonging to $\mathcal{A}$. Here we mean mainly the $(k+1)$th degree homogeneous polynomial p, such as in a lonely algebra (see Definition 8 for 'lonely' on p. 34). Take polynomials $p_j \in \mathcal{A}$ from the algebra and decompose each into monomials (not necessarily belonging to $\mathcal{A}$): $p_j = \sum_{\alpha_j} a_j^{\alpha_j}$, where each $a_j^{\alpha_j}$ is a monomial. Expanding the Wronskian $W_d^k(p_1, \ldots, p_N)$ of the arguments $p_1, \ldots, p_N$, the result will be a linear combination of Wronskians of the monomial arguments $\sum_{\alpha_1,\ldots,\alpha_N} W_d^k(a_1^{\alpha_1}, \ldots, a_N^{\alpha_N})$. The *maximal degree sum* in x^i of $\mathcal{A}$ is defined by taking the maximum over the constituent monomials $a_j^{\alpha_j}$.

Definition 9 (maximal degree sum)**.** Fix the base dimension $d \geqslant 1$ and the differential order $k \geqslant 1$ of the complete generalised Wronskian determinant W_d^k; set $N = \binom{d+k}{k}$. Let $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$ be an N-ary polynomial SH-Lie algebra. Then the *maximal degree sum*, $D_i^{(N)}$, in coordinate x^i over N (linearly independent) polynomials in $\mathcal{A}$ is:

$$D_i^{(N)}(\mathcal{A}) = \max_{P} \max_{A(P)} \sum_{j=1}^{N} \deg_i(a_j^{\alpha_j}), \tag{61}$$

where P consists of all subsets of polynomials $\{p_1, \ldots, p_N\} \subseteq \mathcal{A}$ such that $p_1, \ldots, p_N$ are linearly independent, and, after decomposing each polynomial into constituent monomials $p_j(\boldsymbol{x}) = \sum_{\alpha_j} a_j^{\alpha_j}$ not necessarily belonging to $\mathcal{A}$, put $A(P)$ to be all $\{a_1^{\alpha_1}, \ldots, a_N^{\alpha_N}\}$ such that all the monomials are linearly independent. The function $D_i^{(N)}$ takes values in $\mathbb{N}_{\geqslant 0} \cup \{+\infty\}$.

The properties of a polynomial SH-Lie algebra $\mathcal{A}$ are then characterised by the value of $D_i^{(N)}(\mathcal{A})$ compared with the index-shift down by the complete generalised Wronskian (see Definition 7).

Definition 10. Fix the base dimension $d \geqslant 1$ and the differential order $k \geqslant 1$ of the complete generalised Wronskian determinant W_d^k; set $N = \binom{d+k}{d}$. Recall from Definition 7 that the degree shift in coordinate x^i by W_d^k is $kN/(d+1)$. An N-ary polynomial SH-Lie algebra $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$ with the N-ary bracket given by W_d^k is called:

(*i*) *deficient* in coordinate x^i if the maximal degree sum $D_i^{(N)}$ is strictly smaller than the index shift, that is, $D_i^{(N)}(\mathcal{A}) < kN/(d+1)$;
(*ii*) *exact* in coordinate x^i if $D_i^{(N)}(\mathcal{A}) = kN/(d+1)$;
(*iii*) *abundant* in coordinate x^i if $D_i^{(N)}(\mathcal{A}) > kN/(d+1)$; and
(*iv*) *promising* if the algebra $\mathcal{A}$ is abundant in every coordinate $x^1, \ldots, x^d$.

(The definition of promising is naïve and will likely be revised in the future research.)

This choice of terminology is revealed by the following lemma. Recall the deficiency mechanism for the vanishing of the complete generalised Vandermonde determinant of power

d-tuples of (genuine) monomials: it is *(iii)* from Proposition 9 on p. 32. The two statements follow by applying the degree shift by the Wronskian.

Lemma 19 (deficient vs exact algebra)**.** Fix any base dimension $d \geqslant 1$ and differential order $k \geqslant 1$ of the complete generalised Wronskian determinant W_d^k; set $N = \binom{d+k}{k}$. If an N-ary polynomial SH-Lie algebra $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$, with W_d^k as the N-ary bracket, is *deficient* in at least one coordinate x^i, then *every* choice of N arguments from $\mathcal{A}$ for the Wronskian will yield zero. (Thus $\mathcal{A}$ is 'trivially' closed under W_d^k.)

- If $\mathcal{A} \subseteq \Bbbk[\boldsymbol{x}]$ is exact in coordinate x^i for some $i \in \{1, ..., d\}$, then *every* choice of arguments will result in a polynomial of degree zero in coordinate x^i. (Thus $\mathcal{A}$ cannot be perfect.)

Remark 17. The number of exact coordinates and the minimal number of arguments necessary to attain the maximum in (61) determine the (maximal bounds for) the asymptotic behaviour of iterating the bracket of arguments from $\mathcal{A}$. Indeed, the maximal degree of all polynomial $p(\boldsymbol{x}) \in \mathcal{A}$ in an exact coordinate is bounded by $kN/(d+1)$.

It takes great effort to make such notions precise; relevant results will be announced separately. (Cf. [3] for the notion of an asymptotic dimension of a (SH-)Lie algebra under iterating the bracket.)

Acknowledgements. The first author thanks the Center for Information Technology of the University of Groningen for access to the High Performance Computing cluster, Hábrók. The second author thanks the organisers of the international workshop 'Combinatorics & arithmetic for Physics: special days' – CAP25 (19–21 November 2025 at the IHÉS in Bures-sur-Yvette, France), where this line of research was presented, for a warm atmosphere during the meeting; the second author thanks M. Kontsevich, V. Retakh, and O. Gaber for helpful discussions and advice. The second author thanks also the organisers of the Prague Mathematical Physics seminar, where this work was presented on 7 May 2026, for partial financial support and hospitality. This work has been partially supported by the Bernoulli Institute (Groningen, NL) via project 110135.

References

[1] Brown, F. (2025) Multivariable Vandermonde determinants, amalgams of matrices and Specht modules. *J. Algebra* **678** 253-278 DOI: https://doi.org/10.1016/j.jalgebra.2025.03.040.

[2] Dzhumadil'daev, A. S. (2005) n-Lie structures generated by Wronskians. *Siberian Math. J.* **46** 4 601–612 DOI: https://doi.org/10.1007/s11202-005-0061-7. (*Preprint* `arXiv:math/0202043`)

[3] Gel'fand, I. M. and Kirillov, A. A. (1966) Sur les corps liés aux algèbres enveloppantes des algèbres de Lie (transl. from Russian). *Publications Mathématiques de l'IHÉS* **31** 5–19. Available at URL: http://eudml.org/doc/103872

[4] Kirillov, A. A. and Kontsevich, M. L. (1983) The growth of the Lie algebra generated by two generic vector fields on the line. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, no. 4, 15–20.

[5] Kirillov, A. A., Kontsevich, M. L., and Molev, A. I. (1983) Algebras of intermediate growth. *Akad. Nauk SSSR Inst. Prikl. Mat. Preprint 1983*, no. 39, 19 pp. (Transl. in: *Selecta Math. Soviet.* (1990) **9**:2, 137–153.)

[6] Kiselev, A. V. (2007) On associative Schlessinger–Stasheff algebras and Wronskian determinants. *J. Math. Sci.* **141** 1, 1016–1030
DOI: https://doi.org/10.1007/s10958-007-0028-2. (*Preprint* `arXiv:math/0410185`)

[7] Kiselev, A. V. (2025) Wronskians as N-ary brackets in finite-dimensional analogues of $\mathfrak{sl}(2)$. *J. Phys.*: *Conf. Ser.* **3152** Proc. XXIX Int. conf. 'Integrable Systems & Quantum Symmetries' (7–11 July 2025, CVUT Prague, Czech Republic), Paper 012044. — 8 p. (*Preprint* arXiv:2510.02145)

[8] Kiselev, A. V. (2025) Jacobi identities for Wronskian determinants over multidimension. *Bulgarian J. Phys.* **52**-s1 Quantum Theory & Symmetries, 102–107. (*Preprint* arXiv:2511.03848)

[9] LeVeque, W. J. (1956) "*Topics in number theory*". Vol. **2**. Addison–Wesley Publ. Co., Reading MA, pp.128–134.

[10] Schmidt, W. M. (1980) "*Diophantine approximation*". *Lect. Notes in Math.* vol. **785**. Springer, Berlin, pp.114–150.

[11] Shah K. C. and Kiselev A. V. (2026) The alternating compositions of weighted differential operators yield the weights' Wronskian with which constant ? (*Preprint* arXiv:2605.11137).

[12] Stasheff, J. and Lada, T. (1993) Introduction to SH Lie algebras for physicists. *Internat. J. Theoret. Phys.* **32** 7 1087–1103 DOI: https://doi.org/10.1007/BF00671791. (*Preprint* arXiv:hep-th/9209099)

[13] Stasheff, J. (2019) L-infinity and A-infinity structures. *Higher Structures* **3** 1 292–326 DOI: https://doi.org/10.21136/HS.2019.07

[14] Wolsson, K. (1989) Linear dependence of a function set of m variables with vanishing generalized Wronskians. *Linear Algebra Appl.* **117**, 73–80 DOI: https://doi.org/10.1016/0024-3795(89)90548-X.

Appendix A. Re-proof of the main formula (52) by expansion: the lowest differential order case $k = 1$

Put $p(\boldsymbol{x}) = \boldsymbol{x}^{\vec{n}} = (x^1)^{n_1} \cdots (x^d)^{n_d}$ and $q(\boldsymbol{x}) = \boldsymbol{x}^{\vec{m}} = (x^1)^{m_1} \cdots (x^d)^{m_d}$ over the coordinates $\boldsymbol{x} = (x^1, ..., x^d)$. Let us compute $W^{k=1}_{d\geqslant 2}(p, x^2, x^3, ..., x^d, q)$. By Theorem 7 the result is pq/x multiplied by the complete generalised Vandermonde determinant:

$$\det \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ n_1 & 0 & 0 & 0 & & & & m_1 \\ n_2 & 1 & 0 & 0 & & & & m_2 \\ n_3 & & 1 & 0 & & & & m_3 \\ \vdots & & & & & & & \vdots \\ n_{d-1} & & & & & 1 & 0 & m_{d-1} \\ n_d & & & & & & 1 & m_1 \end{vmatrix}.$$

The expansion by the first column, split into three parts, yields

$$(-)^{d-1}m_1 - n_1 \det \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & & & & m_2 \\ & 1 & & & m_3 \\ & & \ddots & & \vdots \\ & & & 1 & m_d \end{vmatrix} +$$

$$+ \sum_{j=2}^{d} (-)^j n_j \det \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & & & & & & & & & m_1 \\ 1 & & & & & & & & & m_2 \\ & 1 & & & & & & & & m_3 \\ & & 1 & & & & & & & m_4 \\ & & & \ddots & & & & & & \vdots \\ & & & & 1 & 0 & & & & m_{j-1} \\ & & & & & 0 & 1 & & & m_{j+1} \\ & & & & & & & \ddots & & \vdots \\ & & & & & \underbrace{}_{j\text{th}} & & & 1 & m_d \end{vmatrix}, \qquad (62)$$

where the determinant in the second term simplifies to

$$(-)^{d-2}m_2 - (-)^{d-3}m_3 + (-)^{d-4}m_4 - \cdots + (-)^d(-)^{d-d}(m_d - 1) = (-)^{d-2}(m_2 + \cdots + m_d - 1)$$

by expansion over the first column $(1, 1, 0, ...)$, repeated inductively until a 2×2 determinant is reached.

In the third determinant in (62) cycle the first j columns (i.e. until the $(1, ..., 0, 0, ...)^\top$ column), giving a sign $(-)^{j-1}$, and then cycle all rows except the first, giving a sign of $(-)^{d-2}$; the determinant becomes diagonal and equals m_1. When multiplied by $(-)^j n_j$, it yields the jth term $-(-)^{d-1}n_j m_1$ in the sum in (62); the sum hence becomes $-(-)^{d-1}m_1(n_2 + \cdots n_d)$.

Combining the results above turns (62) into

$$(-)^{d-1}\Big\{m_1 + n_1(m_2 + \cdots + m_d - 1) - m_1(n_2 + \cdots + n_d)\Big\} =$$
$$= (-)^{d-1}\{(\deg q - 1)n_1 - (\deg p - 1)m_1\}.$$

This then gives, upon moving the monomial $q(\boldsymbol{x})$ from the last to the first argument,

$$W_{d\geqslant 2}^{k=1}(q, p, x^2, x^3, ..., x^d) = (-)^d(-)^{d-1}\{(\deg q - 1)n_1 - (\deg p - 1)m_1\}\frac{pq}{x^1}$$
$$= \{(\deg p - 1)m_1 - (\deg q - 1)n_1\}\frac{pq}{x^1}.$$

To recover the form in formula in Example 14 (on p. 43) apply the change of coordinates $x^1 \leftrightarrows x^j$ and move the monomial $p(\boldsymbol{x})$ to the position at x^j. The sign changes from both actions cancel identically as they correspond to the same cyclic permutation. This completes the proof. □

Appendix B. Proof of Lemma 10

Proof. In each coördinate x^i we require $\sum_{j=1}^N \deg_i(a_j) = kN/(d+1)$ and summatively $\sum_{j=1}^N \deg(a_j) = kdN/(d+1)$. We take all constants out of the Wronskian, making each a_j monic. Represent $a_j(\boldsymbol{x}) = \boldsymbol{x}^{\vec{\boldsymbol{k}}_j} = (x^i)^{k_j^1} \cdots (x^d)^{k_j^d}$, where $\vec{\boldsymbol{k}}_j = (k_j^1, ..., k_j^d) \in \mathbb{N}_{\geqslant 0}^d$. Define a 'norm' $||\vec{\boldsymbol{k}}_j|| = k_j^1 + \cdots + k_j^d = \deg(a_j)$. The summative degree-sum conditions states that $\sum_{j=1}^N ||\vec{\boldsymbol{k}}_j|| = kdN/(d+1)$. Set $S = \{\vec{\boldsymbol{k}}_1, ..., \vec{\boldsymbol{k}}_N\}$ and define the number of monomials of degree t: $n_t := \#\{\vec{\boldsymbol{k}} \in S : ||\vec{\boldsymbol{k}}|| = t\}$.

We note that if all $||\vec{\boldsymbol{k}}|| \leq k$, then there are only N possible (independent) choices – the $\vec{\boldsymbol{k}}$'s of the standard monomials (cf. Notation 4 on p. 24), i.e. $\vec{\boldsymbol{k}}_j = \vec{\boldsymbol{r}}_j$. We call this set $S_0 = \{\vec{\boldsymbol{r}}_j : 1 \leqslant j \leqslant N\}$ and define $m_t = \#\{\vec{\boldsymbol{k}} \in S_0 : ||\vec{\boldsymbol{k}}|| = t\}$.

Denote the summative degree-sum $T_S = \sum_{\vec{\boldsymbol{k}} \in S} = \sum_{t\geq 0} tn_t$ and $T_{S_0} = \sum_{\vec{\boldsymbol{r}}_j \in S_0} ||\vec{\boldsymbol{r}}_j||$; the numbers must satisfy $T_S = T_{S_0} = kdN/(d+1)$. We proceed to show that if any $||\vec{\boldsymbol{k}}_j|| > k$, then $T_S > T_{S_0}$, giving a contradiction.

(1.) If all $||\vec{\boldsymbol{k}}_j|| > k$, then $T_S = \sum_{t\geqslant 0} tn_t > kN > kN \cdot d/(d+1) = T_{S_0}$, a contradiction.

(2.) Let $t_1, ..., t_\ell > k$ be all degrees $t > k$ for which $n_t > 0$. Suppose for contradiction that such degrees exist, i.e. $\ell > 0$. As $\#S = N$ we have

$$n_0 + n_1 + \cdots + n_k = N - (n_{t_1} + \cdots + n_{t_\ell}) =: N - n. \tag{63}$$

Take any $\vec{\boldsymbol{k}} \in S$ with $||\vec{\boldsymbol{k}}|| = t_j$, $1 \leqslant j \leqslant \ell$. Then some n_s for $s \leqslant k$ must be strictly smaller than m_s. As $s \leqslant k$ we get $t_j - s \geqslant t_j - k$. We call $s = s_j^\nu$, where for each such $||\vec{\boldsymbol{k}}|| = t_j$ we assign an index $1 \leqslant \nu \leqslant n_{t_j}$.

(3.) We now expand the sum

$$\begin{aligned} T_S = \sum_{t\geq 0} tn_t = \sum_{t=0}^{k} tn_t + \sum_{j=1}^{\ell} t_j n_{t_j} = T_{S_0} - \sum_{j=1}^{\ell}\left\{\sum_{\nu=1}^{n_{t_j}} s_j^\nu\right\} + \sum_{j=1}^{\ell} t_j n_{t_j} \\ = T_{S_0} + \sum_{j=1}^{\ell}\left\{t_j n_{t_j} - \sum_{\nu=1}^{n_{t_j}} s_j^\nu\right\}, \end{aligned} \tag{64}$$

where for each j value

$$\begin{aligned} tn_{t_j} - \sum_{\nu=1}^{n_{t_j}} s_j^\nu &= t_j - s_j^1 + t_j - s_j^2 + \cdots + t_j - s_j^{n_{t_j}} \\ &\geqslant t_j - k + \cdots + t_j - k = (t_j - k)n_{t_j} > 0, \end{aligned} \tag{65}$$

hence $T_S \geqslant T_{S_0} + \sum_{j=1}^{\ell}(t_j - k)n_{t_j}$ gives $T_S > T_{S_0}$, a contradiction. Therefore $\ell = 0$. $\square$

Appendix C. Python scripts

SCRIPT 1. Computation of the Wronskian determinant over base dimension d with coordinates $(x, y, z, ...)$ of differential order k; monomial q replaces the standard argument 1 and p replaces any chosen standard argument.

```
import sympy as sp
import itertools as its
import math


x, y, z, w, r, t, u, s  = sp.symbols('x y z w r t u s')
n1,n2,n3,n4,n5,n6,n7,n8 = sp.symbols('n1 n2 n3 n4 n5 n6 n7 n8')
m1,m2,m3,m4,m5,m6,m7,m8 = sp.symbols('m1 m2 m3 m4 m5 m6 m7 m8')
degp, degq = sp.symbols('degp degq')
p, q = sp.symbols('p q')


def difflist(L:list, var):
    """
    Differentiates the arguments in the list L by the differential operator
    composed of var: for instance, the second mixed dervative in x and y is
    given by var=(x,y,).
    """
    out = []
    for i in range(len(L)):
        if type(var)==tuple:out.append(sp.diff(L[i],*var))
        else:
            out.append(sp.diff(L[i],var))

    return out


def wronskian_any(k,ns,ms,xis=[x,y,z,w,r,t,u,s],prepl=(x,),excls=[], printout=True):
    """
    Computes the Wronskian-determinant multibracket of order k of
    [p, ... ,q], where p = (x^i)^{n_i} and q = (x^i)^{m_i} for n_i in ns
    and m_i in ms; the length of ns and ms determine the dimension d.
        By setting prepl to a variable combination other than (x,), the
```

```
    term p may replace prepl rather than x (1st position).
        For incomplete Wronskians, excls defines the list of excluded
    differentials.
    """
    pv = 1
    qv = 1
    if len(ns)!=len(ms):
        print("p,q dimension mismatch!")
        return None

    xis = xis[0:len(ns)]
    for i,ni in enumerate(ns):
        pv *= xis[i]**ni
    for i,mi in enumerate(ms):
        qv *= xis[i]**mi

    row = [pv]
    ders = []
    for ordr in range(1,k+1):
        for comb in its.combinations_with_replacement(xis, ordr):
            if comb not in excls: #exclusions in incomplete Wronskian
                ders.append(comb)

                if comb==(*prepl,):continue #replacement by p
                else:
                    row.append(math.prod(comb))
    row.append(qv)


    M = sp.Matrix([row]+[difflist(row,der) for der in ders])

    det_M = sp.factor(M.det())

    if printout:
        #sp.pprint(M)
        print(f"Wronskian k={k}, d={len(ns)}")
        print(f"p replaces term: {prepl}")
        print(f"excluded derivatives: {excls}")
        print(f"{row} = {det_M}")

    return det_M


# execution example
w = wronskian_any(2,[n1,n2],[m1,m2],prepl=(x,))
```

SCRIPT 2. Computation of the Wronskian of N arbitrary arguments under `row` for any choice of differential order k and base dimension d.

```
import sympy as sp
```

```
import itertools as its
import math


x, y, z, w, r, t, u, s  = sp.symbols('x y z w r t u s')
n1,n2,n3,n4,n5,n6,n7,n8 = sp.symbols('n1 n2 n3 n4 n5 n6 n7 n8')
m1,m2,m3,m4,m5,m6,m7,m8 = sp.symbols('m1 m2 m3 m4 m5 m6 m7 m8')
l1,l2,l3,l4,l5,l6,l7,l8 = sp.symbols('l1 l2 l3 l4 l5 l6 l7 l8')


def difflist(L:list, var):
    out = []
    for i in range(len(L)):
        if type(var)==tuple:out.append(sp.diff(L[i],*var))
        else:
            out.append(sp.diff(L[i],var))

    return out


def wronskian_row(k,d,row,xis=[x,y,z,w,r,t,u,s],printout=True):
    """
    Calculates the Wronskian of the list of N arguments in row.
    """

    xis = xis[0:d]

    ders = []
    coeff = 1
    for ordr in range(1,k+1):
        for comb in its.combinations_with_replacement(xis, ordr):
            ders.append(comb)

            coeff *= math.factorial(comb.count(x))
            coeff *= math.factorial(comb.count(y))

    M = sp.Matrix([row]+[difflist(row,der) for der in ders])

    det_M = sp.factor(M.det())

    if printout:
        #sp.pprint(M)
        print(f"Wronskian k={k}, d={d}; coeff = {coeff}")
        print(f"{row} = {det_M}")

    return det_M


# execution example
```

```
row = [x**l1*y**l2,x**m1*y**m2,x**n1*y**n2]
wr = wronskian_row(1,2,row)
```


Appendix D. Raw computational outputs

Set the base dimension $d = 2$ and the differential order $k = 2, 3, 4, 5$. Compute the Wronskian determinant of arguments $p, y, x^2, xy, ..., y^k, q$, where $p = x^{n_1}y^{n_2}$ and $q = x^{m_1}y^{m_2}$ are arbitrary monomials. The raw outputs of such computations by Script 1, executing the input commands,

```
w = wronskian_any(2,[n1,n2],[m1,m2],prepl=(x,))
w = wronskian_any(3,[n1,n2],[m1,m2],prepl=(x,))
w = wronskian_any(4,[n1,n2],[m1,m2],prepl=(x,))
w = wronskian_any(5,[n1,n2],[m1,m2],prepl=(x,))
```

are given below:[25]

```
Wronskian k=2, d=2
p replaces term: (x,)
excluded derivatives: []
[x**n1*y**n2, y, x**2, x*y, y**2, x**m1*y**m2] = -2*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 2)*(n1 + n2 - 2)*(m1*n2 - m1 - m2*n1 + n1)/x


Wronskian k=3, d=2
p replaces term: (x,)
excluded derivatives: []
[x**n1*y**n2, y, x**2, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**m1*y**m2] = -48*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 3)*(m1 + m2 - 2)*(n1 + n2 - 3)*(n1 + n2 - 2)*(m1*n2 - m1 - m2*n1 + n1)/x


Wronskian k=4, d=2
p replaces term: (x,)
excluded derivatives: []
[x**n1*y**n2, y, x**2, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**4, x**3*y, x**2*y**2, x*y**3, y**4, x**m1*y**m2] = 331776*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 4)*(m1 + m2 - 3)*(m1 + m2 - 2)*(n1 + n2 - 4)*(n1 + n2 - 3)*(n1 + n2 - 2)*(m1*n2 - m1 - m2*n1 + n1)/x


Wronskian k=5, d=2
p replaces term: (x,)
excluded derivatives: []
```

[25]In Python the symbol $\star$ denotes multiplication, whereas $\star\star$ denotes exponentiation (e.g. $x^2 =$ `x**2`).

```
[x**n1*y**n2, y, x**2, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**4, x**3*y, x**2*y**2, x*y**3, y**4, x**5, x**4*y, x**3*y**2, x**2*y**3, x*y**4, y**5, x**m1*y**m2] = 19813556551680*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 5)*(m1 + m2 - 4)*(m1 + m2 - 3)*(m1 + m2 - 2)*(n1 + n2 - 5)*(n1 + n2 - 4)*(n1 + n2 - 3)*(n1 + n2 - 2)*(m1*n2 - m1 - m2*n1 + n1)/x
```

The raw outputs of Script 1, setting the first optional argument to `prepl=(x,x,)`, namely, the monomial p replaces the standard argument $(x)^2$, were

```
Wronskian k=2, d=2
p replaces term: (x, x)
excluded derivatives: []
[x**n1*y**n2, x, y, x*y, y**2, x**m1*y**m2] = x**m1*x**n1*y**m2*y**n2*(2*m1**2*n1*n2 - 2*m1**2*n1 + m1**2*n2**2 - 3*m1**2*n2 + 2*m1**2 - 2*m1*m2*n1**2 + 2*m1*m2*n1 + 2*m1*n1**2 - 2*m1*n1*n2 - m1*n2**2 + 3*m1*n2 - 2*m1 - m2**2*n1**2 + m2**2*n1 + 3*m2*n1**2 - 3*m2*n1 - 2*n1**2 + 2*n1)/x**2

Wronskian k=3, d=2
p replaces term: (x, x)
excluded derivatives: []
[x**n1*y**n2, x, y, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**m1*y**m2] = 48*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 3)*(n1 + n2 - 3)*(2*m1**2*n1*n2 - 2*m1**2*n1 + m1**2*n2**2 - 3*m1**2*n2 + 2*m1**2 - 2*m1*m2*n1**2 + 2*m1*m2*n1 + 2*m1*n1**2 - 2*m1*n1*n2 - m1*n2**2 + 3*m1*n2 - 2*m1 - m2**2*n1**2 + m2**2*n1 + 3*m2*n1**2 - 3*m2*n1 - 2*n1**2 + 2*n1)/x**2


Wronskian k=4, d=2
p replaces term: (x, x)
excluded derivatives: []
[x**n1*y**n2, x, y, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**4, x**3*y, x**2*y**2, x*y**3, y**4, x**m1*y**m2] = -497664*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 4)*(m1 + m2 - 3)*(n1 + n2 - 4)*(n1 + n2 - 3)*(2*m1**2*n1*n2 - 2*m1**2*n1 + m1**2*n2**2 - 3*m1**2*n2 + 2*m1**2 - 2*m1*m2*n1**2 + 2*m1*m2*n1 + 2*m1*n1**2 - 2*m1*n1*n2 - m1*n2**2 + 3*m1*n2 - 2*m1 - m2**2*n1**2 + m2**2*n1 + 3*m2*n1**2 - 3*m2*n1 - 2*n1**2 + 2*n1)/x**2


Wronskian k=5, d=2
p replaces term: (x, x)
excluded derivatives: []
[x**n1*y**n2, x, y, x*y, y**2, x**3, x**2*y, x*y**2, y**3, x**4, x**3*y, x**2*y**2, x*y**3, y**4, x**5, x**4*y, x**3*y**2, x**2*y**3, x*y**4, y**5, x**m1*y**m2] = -39627113103360*x**m1*x**n1*y**m2*y**n2*(m1 + m2 - 5)*(m1 + m2 - 4)*(m1 + m2 - 3)*(n1 + n2 - 5)*(n1 + n2 - 4)*(n1 + n2 - 3)*(2*m1**2*n1*n2 - 2*m1**2*n1 + m1**2*n2**2 - 3*m1**2*n2 + 2*m1**2 - 2*m1*m2*n1**2 + 2*m1*m2*n1 + 2*m1*n1**2 - 2*m1*n1*n2 - m1*n2**2 + 3*m1*n2 - 2*m1 - m2**2*n1**2 + m2**2*n1 + 3*m2*n1**2 - 3*m2*n1 - 2*n1**2 + 2*n1)/x**2
```

ITERATE WRONSKIANS OVER $\mathbb{R}^d$ AS N-ARY BRACKETS ON $\mathbb{R}[x^1,\ldots,x^d]$: THE N-BONACCI NUMBERS BOUND THE HIGHEST TOTAL DEGREES

MARKUSS G. ĶĒNIŅŠ*,* AND ARTHEMY V. KISELEV*,§

ABSTRACT. For the algebra $\mathbb{R}[x^1,\ldots,x^d]$ of polynomials in $d \geqslant 1$ variables, regard the complete generalised Wronskian W_d^k of differential order $k \geqslant 1$ over $\mathbb{R}^d$ as the $N = \binom{d+k}{d}$-ary Lie bracket. Take an N-tuple of polynomials, calculate their Wronskian, and keep re-using the newly-created polynomials to produce more of them. The problem is: how fast do their maximal total degrees grow with the number n of iterations of the bracket? Here enter the N-bonacci numbers defined by the recurrence $F_n^{(N)} = F_{n-1}^{(N)} + \cdots + F_{n-N}^{(N)} \in \mathbb{N}$. We prove that for any choice of the initial arguments, the sequence of highest total degrees $\mathsf{D}_n^{(N)} \geqslant 0$ grows (if at all) asymptotically no faster than the nth N-bonacci number: $\lim_{n\to+\infty}\bigl(\mathsf{D}_n^{(N)}/F_n^{(N)}\bigr) < \infty$. We show that for $d = 1$ and k odd, the highest polynomial degrees do attain the N-bonacci bound.

Introduction. On the algebra $C^\infty(\mathbb{R}^d)$ of smooth functions over $\mathbb{R}^d$ with coordinates $\boldsymbol{x} = (x^1,\ldots,x^d)$, consider the N-linear totally antisymmetric $C^\infty(\mathbb{R}^d)$-valued differential operator $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\cdots\boldsymbol{x}}$ by taking the determinant of $N = \binom{d+k}{d}$ functions and their complete set of derivatives up to differential order $k \geqslant 1$ as the rows.[1] The N-ary operator W_d^k is the *complete generalised Wronskian determinant* of differential order k over $\mathbb{R}^d$.

Example 1. The only Wronskian of two arguments is the operator $W^{0,1} := W^{d=1}_{k=1} = \mathbf{1} \wedge \partial/\partial x$, which, evaluated at two functions $f, g \in C^\infty(\mathbb{R})$, is $W^{0,1}(f,g) = \det\bigl(\begin{smallmatrix} f & g \\ f' & g'\end{smallmatrix}\bigr)$. Yet there are two ternary ($N = 3$) operators, namely the usual Wronskian $W^{0,1,2} := W^{k=2}_{d=1} = \mathbf{1}\wedge\partial_x\wedge\partial_x^2$ and the complete generalised Wronskian $W^{k=1}_{d=2} = \mathbf{1} \wedge \partial_x \wedge \partial_y$; these structures are given by

---

Date: 22 August 2026.

2010 *Mathematics Subject Classification.* 05E15, 11B39, 15A15, secondary 05A10, 05A16, 17B65.

Key words and phrases. Differential polynomial, N-ary Lie bracket, multivariate Wronskian determinant, Fibonacci numbers, N-bonacci numbers, asymptotic growth, growth of degrees, Skolem–Pisot problem.

* *Address*: Bernoulli Institute for Mathematics, Computer Science & Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK Groningen, The Netherlands.

* *Present address*: ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland.

§ Corresponding author. *E-mail*: `A.V.Kiselev@rug.nl` ORCID: 0009-0005-6991-1915.

[1] By convention, the wedge ordering of the derivatives $\partial^{|\sigma|}/\partial\boldsymbol{x}^\sigma$, as they mark the determinant rows, is fixed by the lexicographic ordering of the multi-indices σ: $\varnothing \prec x^1 \prec \ldots \prec x^d \prec x^1x^1 \prec \ldots \prec x^dx^d \prec \ldots \prec \bigl(x^d\bigr)^k$. At every differential order ℓ given in the range $0 \leqslant \ell \leqslant k$, there are $\binom{\ell+(d-1)}{d-1}$ such monomials τ of degree ℓ that do not coincide pairwise, hence as many essentially different derivatives $\partial^{|\tau|}/\partial\boldsymbol{x}^\tau$ of order ℓ. (Note that on $C^\infty(\mathbb{R}^d)$, mixed derivatives coincide.) Over all the orders ℓ up to and reaching k, the sum of binomial coefficients yields the size of Wronskian determinant: $N = \sum_{\ell=0}^k \binom{\ell+d-1}{d-1} = \binom{k+d}{d}$, proven on p. 69.

the determinants

$$W^{0,1,2}(f,g,h)(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} \quad \text{and} \quad W^{k=1}_{d=2}(f,g,h)(x,y) = \begin{vmatrix} f & g & h \\ f_x & g_x & h_x \\ f_y & g_y & h_y \end{vmatrix}, \qquad (1)$$

where f,g,h are suitable functions in d variables, and $f_x := \partial f/\partial x$ and $f_y := \partial f/\partial y$.

Let us keep in mind that the Wronskian determinant $W^{0,1,\ldots,N-1}\bigl(x^{\alpha_1},\ldots,x^{\alpha_N}\bigr)$ of monomials in a single variable x again is a monomial in x, and its coefficient is the Vandermonde determinant (relative to the Wronskian and all the lexicographic orderings in it),

$$\det \mathrm{Van}^{k=N-1}_{d=1}\bigl(\alpha_1,\ldots,\alpha_N\bigr) = \det \mathrm{Van}^{0,1,\ldots,N-1}\bigl(\alpha_1,\ldots,\alpha_N\bigr) = \prod_{1\leqslant i<j\leqslant N} (\alpha_j - \alpha_i).$$

Likewise, the generalised Wronskian determinant of monomials in d variables is again a monomial. In turn, the complete generalised Vandermonde determinants are defined (see [2]) by taking the complete generalised Wronskian determinants $W^{k\geqslant 1}_{d\geqslant 1}$ of $N = \binom{d+k}{d}$ monomials in d variables and then, by inspecting the determinant coefficient preceding the monomial.

Example 2. For the two Wronskian determinants in Eq. (1), we have that $W^{0,1,2}(x^a,x^b,x^c) = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \cdot x^{a+b+c-3}$ and $W^{k=1}_{d=2}(x^{\ell_1}y^{\ell_2}, x^{m_1}y^{m_2}, x^{n_1}y^{n_2}) = \begin{vmatrix} 1 & 1 & 1 \\ \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \end{vmatrix} \cdot x^{\ell_1+m_1+n_1-1}\, y^{\ell_2+m_2+n_2-1}$. The rows of (generalised) Vandermonde's determinants are marked by the monomial multi-indices $1, x, x^2$ and $1, x, y$ that mark the rows of (generalised) Wronskians $W^{0,1,2} = \mathbf{1}\wedge\partial_x\wedge\partial_x^2$ and $W^{k=1}_{d=2} = \mathbf{1}\wedge\partial_x\wedge\partial_y$, respectively.

Appearing in this letter in the context of (generalised) Wronskians, the Fibonacci numbers – and their generalisations, the N-bonacci numbers – serve two tasks: (A) these numbers' asymptotic growth rate yields an upper bound for the degrees of polynomials which *may occur* in the constructions under study; (B) showing up in several examples (parametrised by the dimension $d \geqslant 1$ and differential order $k \geqslant 1$), the Fibonacci or N-bonacci numbers encode the degrees of polynomials which the Wronskian determinants *do produce* from a suitably taken initial set of monomials (by following an explicitly described scheme for iterating the N-ary bracket). In other words, the asymptotic upper bound is attained in many cases.

Let us summarise these results, respectively:

(A): The polynomial degrees cannot grow arbitrarily fast. Let us *iterate* the generalised Wronskian determinant $W^{k\geqslant 1}_{d\geqslant 1}$ of differential order k over $\mathbb{R}^d$ as the $N = \binom{d+k}{d}$-ary bracket on the space of polynomials $\mathbb{R}[x^1,\ldots,x^d]$ in d variables. Fix some N-tuple of monomial arguments to start.[2] Every calculation of the monomials' Wronskian produces the monomial (possibly, previously not available) which can from now on be (re)used as one of the N arguments of the bracket. There can be many schemes to re-use the new monomial, provided its coefficient did not vanish: e.g., the old leftmost argument of the Wronkskian is dropped and the new monomial, now taken with unit coefficient, becomes the last argument. By following all the possible schemes to use the so far attained monomials in the ordered N-tuples of arguments, we inspect the top (i.e. highest appearing) degree of resulting monomials: after $n \in \mathbb{N}$ iterations, how fast does it grow ? We prove that this nth top total degree grows (if at

[2] By convention, the default ordering of monomial arguments is lexicographic: the d variables obey $x^1 \prec \ldots \prec x^d$ and polynomial degrees occur in nondecreasing order: see Eqs. (2–5) in Examples 3–5 on pages 66–68.

all[3]) asymptotically not faster than the nth N-bonacci number. (Here the Fibonacci sequence has window width 2 in the recurrence $F_n = F_{n-1} + F_{n-2}$, while tribonacci numbers have window width 3; the N-bonacci numbers satisfy $F_n^{(N)} = F_{n-1}^{(N)} + \cdots + F_{n-N}^{(N)}$.) The Fibonacci numbers F_n grow asymptotically as $F_n \sim \varphi^n$, with $\varphi = (1+\sqrt{5})/2 \approx 1.618\ldots$ the golden ratio; the N-bonacci numbers $F_n^{(N)}$ grow faster: $\alpha_N \in [\varphi, 2)$ and $F_n^{(N)} \sim \alpha_N^n$.

(B): The polynomial degrees' upper bound can be attained. We investigate whether over arbitrary base dimension $d \geqslant 1$ and for differential order $k \geqslant 1$ of the Wronskian bracket, the upper bound can (or cannot) be attained. In the set-up $d \geqslant 1$ and in the lowest order $k = 1$ in Examples 3 and 4 on p. 66 onwards, we produce a sequence of monomials, with non-zero coefficients, whose degrees in the base variables are the Fibonacci numbers (up to constants). For $d = 1$ and any odd order k (so $N = k+1$ even), in Example 5 on p. 68 we find N monomial arguments and a way to re-use the newly produced monomials such that their (integer) coefficient never vanishes and the upper bound is attained: the monomial degrees equal the N-bonacci numbers plus a constant.[4] (The Fibonacci case is $d = 1 = k$ and the degree shift is $+1$.) We conjecture that for all $N \geqslant 2$ the higher bound α_N^n can asymptotically be attained as $n \to +\infty$.

But what if, for a chosen way to re-use the new monomials in the Wronskian bracket, the Vandermonde determinant standing as the *coefficient* of the output hits zero ? This depends on the initially taken N-tuple of arguments and on the scheme – of re-using the new monomials – that determines the linear recurrence relation for the coefficients. We experimentally find many monomial sequences (over dimensions $d = 1, 2, 3, 4$ and differential orders $k = 1, 2, 3$) such that their coefficients do not vanish as far as $n = 3\,000$ iterations (see Example 6 on p. 71). To determine whether the coefficients *never* vanish for all $n \in \mathbb{N}$ is a particular case of the Skolem–Pisot problem, which is NP-hard.

This letter is structured as follows: in Examples 3–5 we showcase the degree growth as Fibonacci or N-bonacci numbers. These examples introduce the research problem; our main theorem controls the rapidity of growth for the monomial degrees, relating it to the Fibonacci (or N-bonacci) numbers. An immediate corollary – in the context of Kirillov's problem from [3] – bounds the growth of *dimension* for the subspaces generated in the algebra of polynomials by the n-fold iteration of the Wronskian as the N-ary bracket. (For

[3] The polynomial degrees can remain bounded uniformly over all $n \in \mathbb{N}$. For example, let $d = 1$ and $k = 1$, so $N = 2$ (see Example 1), and start the iterations by taking the monomials $a_0 = 1$ and $a_1 = x$: we have that $W_{d=1}^{k=1}(1, x) = 1 = a_1 \in \mathrm{span}_{\mathbb{R}}\langle a_0, a_1\rangle$, thus producing now new degrees in the variable x. Another example over $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ (this time, the Lie algebra $\mathfrak{sl}(2, \Bbbk)$ is perfect, $\mathrm{span}_{\Bbbk}[\mathcal{A}, \mathcal{A}] = \mathcal{A}$, with the Lie bracket $[\cdot, \cdot] = \mathbf{1} \wedge \partial_x$): start with the monomials $e := 1$ and $f := -x^2$, whence put $h := [e, f] = -2x$; the triple of quadratic polynomials $e, h, f \in \Bbbk_2[x]$ satisfies the Chevalley relations $[h, e] = 2e$ and $[h, f] = -2f$. For larger $N > 2$ over the lowest base dimension $d = 1$, the previous example of three-dimensional Lie algebra $\mathfrak{sl}(2, \Bbbk)$ was generalised in [4] to the countable sequence of $(N+1)$-dimensional strongly homotopy Lie algebras $\bigl(\Bbbk_N[x], W_{d=1}^{k=N}\bigr)$, here $\Bbbk = \mathbb{R}$ or $\mathbb{C}$ again. In our recent work [2] we describe – under the natural assumption $\Bbbk_k[x^1, \ldots, x^d] \subseteq \mathcal{A}$ – all such finite-dimensional subalgebras $\mathcal{A} \subsetneq \Bbbk[x^1, \ldots, x^d]$, closed w.r.t. the Wronskian brackets $W_{d\geqslant 1}^{k\geqslant 1}$ on the spaces of polynomials in d variables. In all of these examples, during the iteration of the N-ary bracket by following any schemes for re-using its output, the polynomial degrees cannot exceed the maximal degree of monomials in the algebra $\mathcal{A}$ at hand.

[4] The (Laurent) monomial degree shifts – in the context of Wronskian determinants $W_{d\geqslant 1}^{k\geqslant 1}$ as the N-ary brackets which generalise the Witt algebra structure on $\Bbbk[[x]]$ – are discussed in more detail in [4] (over the base dimension $d = 1$) and in [2] (over higher dimensions $d > 1$).

reader's convenience, in the proposition on p. 70 we recall – and in Appendix B we re-prove – the rapidity of asymptotic growth for the N-bonacci numbers; our bound is asymptotically $\exp(1)/2 \approx 1.359$ times less sharp than Wolfram's in [10] yet our proof is more geometric.) We conclude this letter with a conjecture, itself based on the many examples: there exists an initially taken N-tuple of monomials in d variables such that by iterating the Wronskian bracket and re-using its output, one obtains the sequence of monomials whose degrees grow as fast as the N-bonacci numbers and whose coefficients, given by the generalised Vandermonde determinants, never vanish.

We place this letter in a broader context of (*i*) N-ary Lie structures, in particular strongly homotopy Lie algebras of (non)polynomial functions on $\mathbb{R}^d$ with the (in)complete generalised Wronskian determinants as the N-ary brackets (see [2, 4, 5, 6] and references therein), and (*ii*) Kirillov's problem about Lie algebras of intermediate growth, specifically about the rapidity of growth for the dimensions of vector spaces which are spanned by the output of $n \in \mathbb{N}$ iterations of the Lie bracket, starting with a generic tuple of smooth vector fields[5] (see [3] as well as [2] and references therein).

Let us have three examples: over the various base dimensions $d \geqslant 1$, these examples suggest which polynomial degrees can actually be produced for monomials in d variables by iterating the (generalised) Wronskian determinant of a chosen differential order $k \geqslant 1$.

Example 3 ($d = 1 = k$: Fibonacci)**.** Put $W^{0,1} = \mathbf{1} \wedge \partial/\partial x$ as the Wronskian of first differential order $k = 1$ over $\mathbb{R}^1 \ni x$. Denote by F_n the nth Fibonacci number: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Then the Wronskian of monomials x^{F_n+1} and $x^{F_{n+1}+1}$, whose degrees are successive Fibonacci numbers shifted by $+1$, is

$$W^{0,1}\big(x^{F_n+1}, x^{F_{n+1}+1}\big) = F_{n-1}\, x^{F_{n+2}+1}, \qquad n \geqslant 1. \tag{2}$$

In other words, the result is a monomial with the next Fibonacci number plus shift $+1$ as the degree; the new monomial's coefficient is not zero for $n > 1$.

Notice that, by taking the newly-created monomial as the last subsequent argument and dropping the lowest-degree old first argument, i.e. calculating $W^{0,1}\big(x^{F_{n+1}+1}, x^{F_{n+2}+1}\big)$, one constructs the sequence of monomials – starting from the powers $x^2 = x^{F_2+1}$ and $x^3 = x^{F_3+1}$ – whose degrees are successive Fibonacci numbers.

Example 4 ($d > 1$, $k = 1$: Fibonacci)**.** Let us generalise (2), now over $\mathbb{R}^d \ni (x^1, \ldots, x^d) = \boldsymbol{x}$. Put $W_d^{k=1} = \mathbf{1} \wedge \partial/\partial x^1 \wedge \cdots \wedge \partial/\partial x^d$ as the first-differential-order Wronskian over $\mathbb{R}^d$. Let us choose the $N = d+1$ monomial arguments to iterate the N-ary bracket $W_{d>1}^{k=1}$.

- If $d = 2$ (so we put $x := x^1$ and $y := x^2$ for brevity, cf. Example 2), then $N = 3$, and for any $n \geqslant 5$ we take the three arguments xy, $x^{F_n}y^{F_{n-1}}$, and $x^{F_{n+1}}y^{F_n}$. Their Wronskian determinant equals

$$\begin{vmatrix} xy & x^{F_n}y^{F_{n-1}} & x^{F_{n+1}}y^{F_n} \\ y & F_n\, x^{F_n-1}y^{F_{n-1}} & F_{n+1}\, x^{F_{n+1}-1}y^{F_n} \\ x & F_{n-1}\, x^{F_n}y^{F_{n-1}-1} & F_n\, x^{F_{n+1}}y^{F_n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & F_n & F_{n+1} \\ 1 & F_{n-1} & F_n \end{vmatrix} \cdot x^{F_n+F_{n+1}} \cdot y^{F_{n-1}+F_n}.$$

[5] The upgrade of the binary Lie bracket for vector fields (their differential order $p = 1$ is minimal) to N-ary Lie-type brackets is done by using higher differential order operators on the line $\mathbb{R}$ and their generalisations over $\mathbb{R}^{d\geqslant 1}$ (see [4, 5] and [9]).

The polynomial degrees in the new monomial $x^{F_{n+2}}y^{F_{n+1}}$ are given by the Fibonacci numbers with the same pattern as before: from the arguments' degree pairs (F_n, F_{n-1}) and (F_{n+1}, F_n) we obtain the degree pair (F_{n+2}, F_{n+1}) for x and y. Note that the degrees of either x or y are both strictly increasing and never vanish as n tends to $+\infty$; the total degrees, being the sum of two consecutive Fibonacci numbers, themselves are Fibonacci numbers. The (3×3)-size determinant coefficient of the newly produced monomial expands to[6]

$$\begin{vmatrix} F_n & F_{n+1} \\ F_{n-1} & F_n \end{vmatrix} + \big(F_{n+1} - 2F_n + F_{n-1}\big) = F_{n-3} - (-1)^n \neq 0 \qquad (n \geqslant 5).$$

We conclude that

$$W_{d=2}^{k=1}\big(xy, x^{F_n}y^{F_{n-1}}, x^{F_{n+1}}y^{F_n}\big) = x^{F_{n+2}}y^{F_{n+1}} \cdot \big(F_{n-3} - (-1)^n\big); \tag{3}$$

the coefficient does not vanish[7] for all $n \geqslant 5$. The index of Fibonacci number F_{n-3}, still showing up in the monomial coefficient, retards w.r.t. the index in Eq. (2) over dimension $d = 1$ (now we have $d = 2$); in the future formula (4) over $d \geqslant 3$ the now-retarding Fibonacci term will be lost yet the currently emerging Cassini term will persist.

We notice that in this example over $d = 2$, the Wronskian bracket $W_{d=2}^{k=1}$ of arity $N = d + 1 = 3$ produces the monomials with Φ-bonacci numbers as their partial or total degrees, yet here $\Phi = 2 < N = 3$: the inequality is strict; the bracket's arity N is greater than the window width Φ in the linear recurrence relation $F_{n+1} = F_n + F_{n-1}$. This means that the degree growth is slower than tribonacci's. We expect that the tribonacci growth rate can be attained by using ternary Wronskian brackets.

- If $d \geqslant 3$, we let the first $d-2$ arguments be the unit 1 followed by the $d-3$ monomials (if any, appearing for $d \geqslant 4$) $x^1, \ldots, x^{d-3}$ of total degree $\leqslant 1$ in the variables x^i; next is the degree-three product of the last three coordinates $x^{d-2}x^{d-1}x^d$, and the last two arguments consist of base variables raised to Fibonacci numbers. (For instance, when the base dimension is $d = 3$ with $(x, y, z) \in \mathbb{R}^3$, take $x^{F_n}y^{F_{n-1}}z^{F_{n-2}} = \prod_{i=1}^d (x^i)^{F_{n-i+1}}$ and $\prod_{i=1}^d (x^i)^{F_{n-i+2}}$ as the last two arguments.) Then we claim that

$$W_d^{k=1}\Big(1, x^1, \ldots, x^{d-3}, x^{d-2}x^{d-1}x^d, \prod_{i=1}^d (x^i)^{F_{n-i+1}}, \prod_{i=1}^d (x^i)^{F_{n-i+2}}\Big) = (-1)^{n-d} \prod_{i=1}^d (x^i)^{F_{n-i+3}}, \tag{4}$$

i.e. the result is a monomial whose base variables are raised to the *next* Fibonacci numbers in each variable (compare the index shifts $+1, +2 \to +3$), and the coefficient does not vanish.

Here we produce the sequence of monomials such that the degree in each base coordinate diverges to infinity exponentially fast (as the nth Fibonacci number, with $n \to +\infty$). Again,

[6] Transforming the first term amounts to Cassini's identity:

$$\begin{vmatrix} F_n & F_{n+1} \\ F_{n-1} & F_n \end{vmatrix} = \begin{vmatrix} F_{n-1}+F_{n-2} & F_n+F_{n-1} \\ F_{n-1} & F_{n-1}+F_{n-2} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} F_{n-1} & F_n \\ F_{n-2} & F_{n-1} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}^q \cdot \begin{vmatrix} F_{n-q} & F_{n-q+1} \\ F_{n-q-1} & F_{n-q} \end{vmatrix}.$$

Now let $q := n-1$, thus reaching

$$(-1)^{n-1} \begin{vmatrix} F_1 & F_2 \\ F_0 & F_1 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = (-1)^{n-1}.$$

In the expansion's second term, we use the Fibonacci recurrence relation $F_{n+1} = F_n + F_{n-1}$ (and likewise for the indices n and $n-1$) three times: $F_{n+1} - 2F_n + F_{n-1} = F_{n-3}$.

[7] The coefficient equals $+1$ at $n = 3$ but it vanishes incidentally at $n = 4$. If $n = 1$ or $n = 2$, either Wronskian $W_{d=2}^{k=1}\big(xy, x^1y^0, x^1y^1\big)$ and $W_{d=2}^{k=1}\big(xy, x^1y^1, x^2y^1\big)$ has two repeated arguments, hence it vanishes identically. At $n = 0$ with $F_{-1} = 1$, the Wronskian is $W_{d=2}^{k=1}\big(xy, x^0y^1, x^1y^0\big) = xy$.

the Fibonacci-fast degree growth is slower than N-bonacci's, where $N = d + 1 \geqslant 4$ is the arity of the Wronskian bracket $W^{k=1}_{d\geqslant 3}$.

Proof of equality (4). By [2, Th. 7 on p. 12] about the Wronskian of monomial arguments, the value of the Wronskian in (4) is the monomial whose degree in coordinate x^i is $(1 + F_{n-i+1} + F_{n-i+2}) - 1 = F_{n-i+3}$ and whose coefficient is the (generalised Vandermonde) determinant

$$\det \left|\begin{array}{c|ccc|ccc} 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & \cdots & 0 & 0 & F_n & F_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & F_{n-d+4} & F_{n-d+5} \\ \hline 0 & 0 & \cdots & 0 & 1 & F_{n-d+3} & F_{n-d+4} \\ 0 & 0 & \cdots & 0 & 1 & F_{n-d+2} & F_{n-d+3} \\ 0 & 0 & \cdots & 0 & 1 & F_{n-d+1} & F_{n-d+2} \end{array}\right| = \begin{vmatrix} 1 & F_{n-d+3} & F_{n-d+4} \\ 1 & F_{n-d+2} & F_{n-d+3} \\ 1 & F_{n-d+1} & F_{n-d+2} \end{vmatrix},$$

which simplifies to Cassini's identity, $F^2_{n-d+1} - F_{n-d}\,F_{n-d+2} = (-1)^{n-d}$, as required. □

Example 5 ($d = 1$, $k \geqslant 1$: N-bonacci)**.** Let us return to base $\mathbb{R}^1 \ni x$ and see if faster – than Fibonacci's – growth of the monomial degrees is possible. Take an odd differential order $k \in \{1, 3, 5, \ldots\}$, so that the Wronskian of even arity $N = k + 1$ becomes $W^{0,1,\ldots,N-1} = \mathbf{1} \wedge \partial_x \wedge \cdots \wedge \partial_x^{N-1}$. The N-bonacci numbers $F_n^{(N)}$ are given by $F_0^{(N)} = \cdots = F_{N-2}^{(N)} = 0$, $F_{N-1}^{(N)} = 1$, and $F_n^{(N)} = F_{n-1}^{(N)} + \cdots + F_{n-N}^{(N)}$, i.e. summation is taken over the preceding N numbers in the sequence. Then the Wronskian of N monomials – whose degrees are N successive N-bonacci numbers plus the shift $N/2$ – is the monomial with the next N-bonacci number in the degree,

$$W^{0,1,\ldots,N-1}\bigl(x^{F_n^{(N)}+N/2}, \ldots, x^{F_{n+N}^{(N)}+N/2}\bigr) = \prod_{0\leqslant i<j<N} \bigl(\underbrace{F_{n+j}^{(N)} - F_{n+i}^{(N)}}_{>0}\bigr) \cdot x^{F_{n+N+1}^{(N)}+N/2}; \qquad (5)$$

the monomial's coefficient does not vanish[8] for $n \geqslant N$.

We thus produce a sequence of monomials whose degrees grow faster than the Fibonacci numbers; their growth is again exponential, but asymptotically slower than 2^n; see the Proposition on p. 70 about their growth. (The case 2^n corresponds to the ∞-bonacci numbers, where summation is taken over *all* previous terms.)

Research problem. Take the complete generalised Wronskian $W_d^k = \mathbf{1} \wedge \partial_{\boldsymbol{x}} \wedge \cdots \wedge \partial^k_{\boldsymbol{x}\ldots\boldsymbol{x}}$ of differential order $k \geqslant 1$ over $\mathbb{R}^{d\geqslant 1} \ni \boldsymbol{x}$ as the N-ary Lie bracket, $N = \binom{d+k}{d}$. Pick some set of initially taken polynomial arguments from $\mathbb{R}[x^1, \ldots, x^d]$, denoting their span by $\mathcal{A}_0$, and put D_0 to be the highest total polynomial degree among them. Plug the polynomials from $\mathcal{A}_0$ into the Wronskian, obtaining (possibly new) polynomials, the span of which we denote by $\mathcal{B}_1$. Put $\mathcal{A}_1 := \operatorname{span}(\mathcal{A}_0 \cup \mathcal{B}_1)$. Now denote by D_1 the highest total degree among the polynomials from $\mathcal{A}_1$. Repeat this process of iterating the N-ary bracket using polynomials from $\mathcal{A}_1$, obtaining the (sub-)spaces of polynomials $\mathcal{A}_2, \mathcal{A}_3, \ldots$ and the highest total degrees $\mathsf{D}_2, \mathsf{D}_3, \ldots$ of polynomials among them; let us introduce the notation $\mathsf{D}_n = \mathsf{D}_n^{(N)}$ to emphasise the degrees' dependence on the arity N.

[8]When $n = N - 1$, there is a repeated argument $x^{1+N/2}$, hence the Wronskian determinant vanishes.

The problem is to determine how fast the sequence $\{\mathsf{D}_n^{(N)}\}_{n\in\mathbb{N}}$ grows (if at all) as $n \to +\infty$. In particular, *what is the fastest possible growth of* $\{\mathsf{D}_n^{(N)}\}_{n\in\mathbb{N}}$ *among all choices of the initial arguments,* $\mathcal{A}_0 \subsetneq \mathbb{R}[x^1, \ldots, x^d]$?

Our main result is an upper bound for the asymptotic growth of the (poly- or) monomials' degrees $\{\mathsf{D}_n^{(N)}\}_{n\in\mathbb{N}}$.

Theorem. *Let the base dimension* $d = \dim \mathbb{R}^d \geqslant 1$ *and differential order* $k \geqslant 1$ *of the complete generalised Wronskian* W_d^k *be arbitrary; put* $N = \binom{d+k}{d}$. *For any choice of initial arguments for the Wronskian, the sequence of highest total degrees* $\{\mathsf{D}_n^{(N)}\}$ *(and the highest degree in each variable)* grows (if at all) asymptotically no faster than the N-bonacci numbers.

(The proof is given in Appendix A.)

Now we can bound the growth in dimension of the subspaces $\mathcal{A}_0$, $\mathcal{A}_1$, ...; the growth in dimension, $\dim(\mathcal{A}_n)$, is Kirillov's problem [3] extended to the (polynomial) case of N-ary Lie algebras $\mathcal{A} \subseteq \Bbbk\big[x^1, \ldots, x^d\big]$ (see [2] for the answer in the base case — for certain $\mathfrak{sl}(2,\Bbbk)$-line algebras that do not grow at all).

Corollary (towards Kirillov's problem)**.** For any dimension $d \geqslant 1$ and differential order $k \geqslant 1$, so that $N = \binom{d+k}{d}$, and any initially taken polynomial arguments for the Wronskian, the dimensions $\dim(\mathcal{A}_n)$ of the subspaces of polynomials created after $n > 0$ iterations of the N-ary bracket grow (if at all) no faster than the following asymptotic bound, as $n \to +\infty$:

$$\log \dim(\mathcal{A}_n) \lesssim n \cdot \log(\alpha_N^d) + O(1) < n \cdot \log(2^d) + O(1), \tag{6}$$

where $\alpha_N = \lim\limits_{n\to+\infty} (F_{n+1}^{(N)}/F_n^{(N)}) \in [\varphi, 2)$ describes the asymptotic growth of the N-bonacci numbers, see the proposition on the next page. (Here $\varphi = (1+\sqrt{5})/2 \approx 1.618\ldots$ is the golden ratio.)

Proof (of the Corollary). The dimension of the space of polynomials in d variables up to degree m is given[9] by the binomial coefficient $\binom{d+m}{d} = m^d/d! + O\big(m^{d-1}\big)$. As the degrees of all monomials created after $n \in \mathbb{N}$ iterations of the Wronskian N-ary bracket do not exceed $\mathsf{D}_n^{(N)}$, it follows that

$$\dim(\mathcal{A}_n) \leqslant \binom{d+\mathsf{D}_n^{(N)}}{d} = \big(\mathsf{D}_n^{(N)}\big)^d/d! + O\Big(\big(\mathsf{D}_n^{(N)}\big)^{d-1}\Big).$$

As the highest degrees $\mathsf{D}_n^{(N)}$ grow asymptotically no faster than the N-bonacci numbers, $\log \mathsf{D}_n^{(N)} \lesssim \log \alpha_N^n + O(1)$ as $n \to +\infty$, we conclude that

$$\log \dim(\mathcal{A}_n) \lesssim d \cdot \log\big(\mathsf{D}_n^{(N)}\big) + O(1) \lesssim n \cdot \log(\alpha_N^d) + O(1) < n \cdot \log(2^d) + O(1),$$

[9] The number of linearly independent monomials τ of degrees $\leqslant k$ in d variables x^1, ..., x^d, naturally equal to the number of different (on $C^\infty(\mathbb{R}^d)$) partial derivatives $\partial^{|\tau|}/\partial \boldsymbol{x}^\tau$ of orders $\leqslant k$, is $N = \binom{k+d}{d}$. The formula $\sum_{\ell=0}^{k} \binom{\ell+d-1}{d-1} = \binom{k+d}{d}$, expressing the dimension $\dim \Bbbk_k\big[x^1, \ldots, x^d\big]$, hence the size of the complete generalised Wronskian determinant W_d^k, is proven by induction over ℓ. The base of induction is $\binom{0+d-1}{d-1} = 1 = \binom{d}{d}$. The inductive step uses the binomial identity $\binom{m}{d} + \binom{m}{d-1} = \binom{m+1}{d}$; here m and d are positive integers. Running the differential order ℓ up, after k steps we accumulate $\binom{k+d}{d}$ partial derivatives (in d variables) of all nonnegative orders not exceeding k. This proves the claim.

as claimed.[10] □

To provide more details about the limit values α_N, let us recall some basic facts about linear integer recurrences with constant coefficients. Define the integer sequence $y_n \in \mathbb{Z}$ recursively by $y_n = a_1 y_{n-1} + \cdots a_N y_{n-N} + b$, where $a_1, \ldots, a_N, b \in \mathbb{Z}$ are constants, and the sequence begins from some initial datum $y_1^0, \ldots, y_N^0$; the sequence is homogenised, $x_n = a_1 x_{n-1} + \cdots + a_N x_{n-N}$, by setting $x_n := y_n - y^*$ if $y^* := b/(1 - a_1 - \cdots - a_N)$ exists. In particular, the sequence $y_n = y_{n-1} + \cdots + y_{n-N} + b$ can be homogenised, yielding the N-bonacci recurrence relation $x_n = x_{n-1} + \cdots + x_{n-N}$; its closed-form solution is $x_n = c_1\lambda_1^n + \cdots + c_N\lambda_N^n$, where $c_i \in \mathbb{C}$ are constants (that depend on the initial data) and each $\lambda_i \in \mathbb{C}$ is a root of the characteristic polynomial

$$p(\lambda) = \lambda^N - \lambda^{N-1} - \cdots - \lambda - 1. \tag{7}$$

The task is to determine the roots of $p(\lambda)$. We claim that $N - 1$ roots lie within the complex unit disk, whereas the Nth root, denoted by α_N, lies in the interval $(1, 2) \subsetneq \mathbb{R}$, see Fig. 1 on p. 75 below. If the coefficient $c_N =: C$ preceding α_N (and determined by the initial datum) is non-zero, the sequence $\{x_n\}$ is, asymptotically, $x_n \simeq C\alpha_N^n$, which diverges exponentially to $+\infty$ (if $C > 0$) as $n \to +\infty$.

Proposition ([10, Lemma 3.6]: growth of N-bonacci numbers)**.** The N-bonacci numbers $F_n^{(N)}$, given by the recurrence $F_n^{(N)} = F_{n-1}^{(N)} + \cdots + F_{n-N}^{(N)}$ from $F_0^{(N)} = \cdots = F_{N-2}^{(N)} = 0$ and $F_{N-1}^{(N)} = 1$, grow asymptotically as $F_n^{(N)} \simeq C\alpha_N^n$, where Wolfram's lower bound is $2(1 - 2^{-N}) < \alpha_N < 2$ and $C > 0$. (All other roots λ_i of characteristic polynomial (7) satisfy the bounds $3^{-1/N} < |\lambda_i| < 1$, i.e. they lie within the annulus close to the boundary of the complex unit disk.)

- The numbers α_N are monotonically increasing w.r.t. N as $\varphi = \alpha_2 < \alpha_3 < \cdots < \alpha_\infty = 2$.

In Appendix B we re-prove this fact using Rouché's theorem [8, § 10.43 (b)] from complex analysis, obtaining a slightly smaller than Wolfram's, but asymptotically equal (as $N \to \infty$) lower bound for $\alpha_N = 2(1 - \varepsilon_N)$: our bound is $\varepsilon_N < \delta_N := \frac{1}{2}(1 + 1/N)^N 2^{-N}$, which yields the limit $\lim_{N\to+\infty}(\delta_N/2^{-N}) = e/2 \approx 1.359$ that compares our bound with Wolfram's. Our approach is very elementary and geometric; it could be refined further to find a sharper bound.

Remark 1 (Pisot–Vijayaraghavan number). The number α_N is called a Pisot–Vijayaraghavan (PV) number as it is the only real root $\alpha_N > 1$ of the monic irreducible polynomial $p(\lambda) \in \mathbb{Z}[\lambda]$ whose all other roots lie within the complex unit disk. It is known that as $n \to +\infty$, the powers α_N^n of the PV numbers are 'almost integers' whose distance to the nearest integer tends to 0 as fast as $\exp(-n)$. So, to get (a precise approximation of) the integer $F_n^{(N)} = C\alpha_N^n + (\exp \text{small terms})$, it suffices to know the constants C and α_N and the index $n \in \mathbb{N}$. Conversely, for large n (hence $F_n^{(N)} \gg 1$), one easily gets (a good approximation of C and

---

[10] At this point, the next step along this line of research would be a sufficient number of (computationally costly) examples revealing the actual rapidity of growth for the dimensions $\dim(\mathcal{A}_{n\geqslant 1})$, starting from generic $(M \geqslant N)$-tuples of initially taken polynomials $a_i \in \mathcal{A}_0$. In particular, the upper bound $\dim(\mathcal{A}_n) \leqslant \text{const}(d) \cdot \exp(\text{const}(d) \cdot n^1)$, de facto available from (6), might far exceed the actual asymptotic growth rates — due to the integer power n^1 under the exponent (in contrast with the fractional power $n^{2/3}$ in the seminal work [3] over the base dimension $d = 1$). From the proof of asymptotic bound (6) we now see that the dimension $d \geqslant 1$ shows up explicitly in either constant around the exponential function of the upper bound.

then) the index n for a given N-bonacci number $F_n^{(N)}$.

Remark 2 (upper bound satisfied). Notice that in Examples 3 and 5 – corresponding to the Wronskians over $\mathbb{R}^1$ of odd differential order k – we have attained the upper bound for the growth in degree: the monomials' degrees *are* the N-bonacci numbers, and their coefficients never vanish. Simultaneously, it is an **open problem** whether the bound can be attained – with nonzero coefficient – over higher base dimension $d = \dim \mathbb{R}^d > 1$ and any differential order $k \geqslant 1$. (In Example 4 the coefficients are seen to be nonzero both in (3) over $d = 2$ and in (4) over $d \geqslant 3$, yet these were specific to the differential order $k = 1$ in the Wronskian bracket W_d^k.)

Example 6 (computer algebra experiments)**.** Using the computer algebra (`SymPy`) script in Appendix C of [2] – for calculating the complete generalised Wronskian determinants of monomial arguments, – we made 32 experiments over base dimensions $d = 1, 2, 3, 4$ and differential orders $k = 1, 2, 3$ by choosing an initial N-tuple of monomial arguments and iterating the bracket: re-inserting the newly-created monomials as arguments; these computations are reported in the ancillary file.[11]

Sample scheme. Fix the base dimension d of $\mathbb{R}^d \ni \boldsymbol{x} = (x^1, \ldots, x^d)$ and differential order k of the Wronskian determinant W_d^k. Take some $N = \binom{d+k}{d}$-tuple $(m_1(\boldsymbol{x}), \ldots, m_N(\boldsymbol{x}))$ of monomial arguments $m_n(\boldsymbol{x}) \in \mathbb{R}[x^1, \ldots, x^d]$ and denote by $b_n := \deg_i(m_n)$ their degrees in some coordinate x^i; set up the scheme of iteration for the indices $n \geqslant N+1$ by $m_n(\boldsymbol{x}) := W_d^k(m_{n-N}, \ldots, m_{n-1})$. By [2, Theorem 7 on p. 12], the monomials' degree b_n in each base variable satisfies the linear recurrence relation $b_n = b_{n-1} + \cdots + b_{n-N} - kN/(d+1)$, which, upon homogenisation, becomes the recurrence relation for the N-bonacci numbers. The monomials' coefficients satisfy their own (different, if any) recurrence relation.

Sample bracket. For instance, deploy the Wronskian $W_{d=2}^{k=1} = \mathbf{1} \wedge \partial/\partial x \wedge \partial/\partial y$ of first order over $\mathbb{R}^2 \ni (x, y)$ as the ternary ($N = 3$) bracket; take as the three arguments the (monic) monomials $x^{a_n} y^{a_{n+1}}$, $x^{a_{n+1}} y^{a_{n+2}}$, and $x^{a_{n+2}} y^{a_{n+3}}$, where the integers a_n are given by the ternary recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3} - 1$ starting from a suitably chosen initial datum (a_1, a_2, a_3). The Wronskian of these three monomials is the monomial $x^{a_{n+3}} y^{a_{n+4}}$ with coefficient c_n. We verified using computer algebra that the coefficients $\{c_n\}_{n \in \mathbb{N}}$ satisfy the recurrence relation $c_n = -c_{n-1} - c_{n-2} + c_{n-3}$, whence the integers c_n oscillate between large positive and negative values. It is unknown whether an initial datum (a_1, a_2, a_3) exists such that the coefficients c_n never vanish. We found many choices of initial data such that $c_n \neq 0$ up to $n \approx 3\,000$.

Remark 3 (Skolem–Pisot problem). Given a linear recurrence relation for $z_n \in \mathbb{Z}$ and an initial datum, the Skolem–Pisot problem is to determine whether the sequence of integers

[11] The format of each entry in the output is this: the base dimension $d \geqslant 1$, the differential order $k \geqslant 1$, the arity $N = \binom{d+k}{d}$ of the complete generalised Wronskian; after the N-tuple of initially taken monomial arguments, we describe the scheme: which of the arguments are fixed and how the newly produced monomials are re-used. In this setting, we write the starting 50 coefficients of the monomials which are produced by iterating the Wronskian W_d^k. We report the overall number of iterations actually run in the example at hand and finally, the – sometimes, huge – coefficient obtained in the last iteration. (It is the threshold of circa 10 000 iterations on or near which the coefficients get so big that the software hits the maximal operationally admissible number of digits in a decimal number. This is why the actual number of reported iterations is less than 10 000.)

$\{z_n\}_{n\in\mathbb{N}}$ ever hits zero: $z_n = 0$. If zeros exist, it is known (the Skolem–Mahler–Lech Theorem, see [1]) that the set of indices $n \in \mathbb{N}$ for which $z_n = 0$ consists of the union of a (possibly empty) finite set with a set of arithmetic progressions. In general, it is NP-hard to decide whether the sequence $\{z_n\}_{n\in\mathbb{N}}$ will ever hit a zero [1].

Conjecture. Take the Wronskian W_d^k of any differential order $k \geqslant 1$ over any base dimension $d = \dim \mathbb{R}^d \geqslant 1$ as the $N = \binom{d+k}{d}$-ary bracket on the space of polynomials $\mathbb{R}[x^1, \ldots, x^d]$. We conjecture that there exists an N-tuple of monomials $(m_1(\boldsymbol{x}), \ldots, m_N(\boldsymbol{x})) \in \mathbb{R}[x^1, \ldots, x^d]$ such that the scheme $m_n(\boldsymbol{x}) := W_d^k(m_{n-N}, \ldots, m_{n-1})$ to iterate the bracket over $n \in \mathbb{N}$ yields the monomials $m_n(\boldsymbol{x})$ whose degrees (total or in some, or in every variable) grow asymptotically as fast as the N-bonacci numbers (by construction) and *whose coefficients are never zero:* $m_n(\boldsymbol{x}) \not\equiv 0$ *for all* $n \in \mathbb{N}$.

Open problem. Over arbitrary dimensions $d \geqslant 1$ and differential orders $k \geqslant 1$, from [2] we know countably many examples $\mathcal{A}(d,k)$ of finite-dimensional N-ary Lie algebras, $\Bbbk_k[x^1, \ldots, x^d] \subseteq \mathcal{A} \subsetneq \Bbbk[x^1, \ldots, x^d]$: their dimension does not grow in the course of iterating the Wronskian bracket. The next task is to find regular classes (for $d, k \in \mathbb{N}$) of the subspaces $\mathcal{A}_0 \subsetneq \Bbbk[x^1, \ldots, x^d]$ such that with these initially taken arguments, the highest degree growth is linear, or faster that linear but at most Fibonacci's, or faster than Fibonacci still at most tribonacci-fast, etc. Do there exist initial sets $\mathcal{A}_0$ providing the polynomial degrees' intermediate growth: between no growth and linear, or between linear and Fibonacci, and so on ?

Conclusion. The asymptotic bound (see our theorem on p. 69) of polynomial degrees that are obtained by iterating the Wronskian W_d^k as the $N = \binom{d+k}{d}$-ary bracket on $\mathbb{R}[x^1, \ldots, x^d]$ implies the asymptotic bound in Eq. (6) for the dimensions of vector subspaces $\mathcal{A}_n \subsetneq \mathbb{R}[x^1, \ldots, x^d]$ spanned after the nth iteration. One would expect that, whatever be the schemes to re-use the newly created monomials, far not all of the lower-degree monomials – within the degree bound – will effectively be produced and available after the nth step (if ever at all); if so, the actual dimensions of subspaces $\mathcal{A}_n$ will tend to be smaller. We pose the problem of improving – i.e. lowering – the upper bound upon the growth rate for the subspace dimensions $\dim \mathcal{A}_n$ in the (strongly homotopy Lie) algebras of polynomials on $\mathbb{R}^d$.

Our present choice to first study the polynomials – as objects' coefficients – made the research problem clearly distinct from Kirillov's original formulation in [3] for generic vector fields with smooth coefficients on the line $\mathbb{R}$. Unlike the generic tuples of initially taken functions $f_1, \ldots, f_{r\geqslant N} \in C^\infty(\mathbb{R}^d)$ which, by definition, are not constrained by any differential-algebraic relations,[12] the polynomials in $\mathcal{A}_0$ have their degrees bounded uniformly by $\mathsf{D}_0^{(N)}$. (Naturally, all sufficiently high order derivatives of every polynomial function vanish identically, to begin with.) The risk to encounter a slower rapidity of the degree growth (if it does, and apart from reproduction of the earlier reached monomials) is, in the case of $\mathbb{R}[x^1, \ldots, x^d] \subsetneq C^\infty(\mathbb{R}^d)$, due to the abrupt termination of those monomial sequences which hit the zero coefficient. The problem thus doubles, now incorporating Kirillov's question about the dimensions $\dim \mathcal{A}_n$ and the Skolem–Pisot problem about the coefficients (of the monomials that span $\mathcal{A}_n$). Our conjecture, see above, means that the growth does persist for

[12] Each Jacobi(-type) identity of the (strongly homotopy) Lie algebra constrains the structure, with respect to which the identities are quadratic (see [5, 6] and many references therein), yet none of the Jacobi identities restricts the set of its arguments. For example, see [4, 5] for the Jacobi identities which either of the ternary brackets in (1) satisfies for *arbitrary* 5-tuples of smooth functions on the respective base space $\mathbb{R}^d$.

generic tuples of initially taken polynomials on $\mathbb{R}^d$. It would be interesting to compare the observed growth rates for the case of polynomials and their N-ary brackets $W^{k\geqslant 1}_{d\geqslant 1}$ with the intermediate growth rate for the case of smooth functions as vector field coefficients and their binary Lie bracket W^1_1 in [3].

Acknowledgements. The first author thanks the Center for Information Technology of the University of Groningen for access to the High Performance Computing cluster, Hábrók. The second author thanks M. Kontsevich for suggesting to extend the research problem in [3] from binary Lie brackets of vector fields to the N-ary Lie-type brackets given on (the algebra of coefficients) $C^\infty(\mathbb{R}^d) \supsetneq \mathbb{R}[x^1,\ldots,x^d]$ by the generalised Wronskians $W^{k\geqslant 1}_{d\geqslant 1}$ of differential order k, and for stimulating discussions at the IHÉS.

Statements and Declarations.

Authors' contribution statement. Both authors contributed equally to this work.

Data availability statement. The data supporting the findings in Example 6 are available as text file at arXiv:2607.26039v2.

Conflict of interests statement. On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

[1] Blondel V. D., Portier N. (2002) The presence of a zero in an integer linear recurrent sequence is NP-hard to decide. *Linear Algebra Its Appl.* **351–352**, 91–98.

[2] Ķēniņš M. G., Kiselev A. V. (2026) Explicit classs of finite-dimensional polynomial algebras with Wronskians over $\mathbb{R}^d$ as N-ary Lie brackets: Beyond $\mathfrak{sl}(2)$. *Preprint* `arXiv:2605.27305` [math.RA], 44 p.

[3] Kirillov A. A., Kontsevich M. L. (1983) The growth of the Lie algebra generated by two generic vector fields on the line. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.*, no. 4, 15–20.

[4] Kiselev A. V. (2005) On associative Schlessinger–Stasheff algebras and Wronskian determinants, *Fundam. Prikl. Matem.* **11**:1, 159–180; English transl. *J. Math. Sci.* (2007) **141**:1, 1016–1030. (*Preprint* `arXiv:math.RA/0410185`)

[5] Kiselev A. V. (2025) Wronskians as N-ary brackets in finite-dimensional analogues of $\mathfrak{sl}(2)$, *J. Phys.: Conf. Ser.* **3152**, Paper 012044, 1–8. (*Preprint* `arXiv:2510.02145` [math.RA])

[6] Kiselev A. V. (2025) Jacobi identities for Wronskian determinants over multidimension, *Bulgarian J. Phys.* **52**-s1 Quantum Theory & Symmetries, 102–107. (*Preprint* `arXiv:2511.03848` [math.RA])

[7] Roots of $x^n - x^{n-1} - \cdots - x - 1$. *MathOverflow* (2024). URL: https://mathoverflow.net/q/332421

[8] Rudin W. (1966) "*Real and Complex Analysis*" (2nd ed), McGraw–Hill Book Co.

[9] Shah K. C., Kiselev A. V. (2026) The alternating compositions of weighted differential operators yield the weights' Wronskian with which constant ? *Preprint* `arXiv:2605.11137` [math.CO], 22 p.

[10] Wolfram D. A. (1998) Solving generalised Fibonacci recurrences. *Fibonacci Quarterly* **36**, 2, 129–145.

Appendix A. Proof of the main Theorem

Proof (of the main Theorem). It suffices, by multilinearity, to assume that the set of initial arguments consists of monomials. By [2, Theorem 7 on p. 12], the Wronskian of monomial arguments $p_1(\boldsymbol{x}),\ldots,p_N(\boldsymbol{x})$ is the monomial $p_1\cdots p_N / \prod_{i=1}^d (x^i)^{kN/(d+1)}$ preceded by the constant given by the generalised Vandermonde determinant of the monomials' degrees in each variable. We find an upper bound for the degree in each variable (and hence the total degree) by presently assuming that this constant never vanishes.

Iterate the Wronskian with the non-vanishing constant n times and consider a monomial $q_n(\boldsymbol{x})$ with the highest total degree $\mathsf{D}_n^{(N)} = \deg q_n(\boldsymbol{x})$; it was produced by some sequence of re-used monomial arguments. If the newly-created argument was not used immediately in the next iteration, the growth was necessarily not faster than maximal. Let us consider which of the old arguments to drop in place of the new one; by the mechanism of degree growth, dropping an argument with the smallest degree results in the monomial with the highest degree. The degrees a_n in each base variable of q_n satisfy the recurrence $a_n = a_{n-1}+\cdots+a_{n-N}-kN/(d+1)$. By the properties of linear recurrences, after homogenisation, the closed form solution is $a_n = a^* + C\alpha_N^n + \sum_{j=1}^{N-1} c_j\lambda_j^n$, where $\alpha_N > 1$ and $a^*, C \in \mathbb{R}$, $c_j \in \mathbb{C}$ are constants, and where all the values λ_j lie in the complex unit disk: $|\lambda_j| < 1$. (See the Proposition on p. 70 and the accompanying discussion.)

Hence $\mathsf{D}_n^{(N)} = a_n^{(1)} + \cdots + a_n^{(d)} = (C^{(1)} + \cdots + C^{(d)})\alpha_N^n + d\cdot a^* + \sum_{i,j} c_j^{(i)}\lambda_j^n$, where $C^{(i)}$ is the constant C for the recurrence of degrees a_n in base variable x^i. Therefore, assuming for the uppper bound that an initial condition exists with $C^{(i)} \neq 0$ for all i, we have $\lim_{n\to+\infty}\bigl(\mathsf{D}_n^{(N)}/F_n^{(N)}\bigr) = C^{(1)} + \cdots + C^{(d)} < \infty$, completing the proof. □

APPENDIX B. RE-PROOF OF THE PROPOSITION AND COMPARISON OF THE BOUNDS

Re-proof (of the Proposition). The characteristic polynomial $p(z)$ of the N-bonacci recurrence is $p(z) = z^N - z^{N-1} - \cdots - z - 1$ (cf. Eq. (7)). Let us analyse where its roots are located in the complex plane. (For example, the numerical values of the roots of $p(z)$ and their lower bounds are shown in Fig. 1 for $N = 2$ (Fibonacci), $N = 3$ (tribonacci), $N = 6$, and $N = 11$.) This reasoning consists of two parts.

1. To demonstrate that $N - 1$ roots of $p(z)$ lie within the complex unit disk, we recall the argument from [7]. Rewrite the polynomial $P(z) := (1 - z)p(z) = z^{N+1} - 2z^N + 1 = (z^{N+1}+1)+(-2z^N)$ as the sum of two holomorphic functions on $D_{r>1} = \{z \in \mathbb{C} : |z| < r\}$. On the boundary $\partial D_r = \{z : |z| = r\}$ we have (for $r > 1$ close to 1), by the triangle inequality (which asserts the first inequality below),

$$|z^{N+1} + 1| \leqslant r^{N+1} + 1 < 2r^N = |-2z^N|; \tag{8}$$

as $P(1) = 0$ and $P'(1) = 1 - N$ is negative, the polynomial $P(x)$ stays negative on some one-sided interval $\bigl(1, r+\Delta r_{>0}\bigr)$ close to 1 that includes $r > 1$, whence the second inequality above. Now, as ∂D_r is a simple curve in $\mathbb{C}$ and we have inequality (8), it follows by Rouche's theorem [8, § 10.43 (b)] that $(z^{N+1} + 1) + (-2z^N)$ and $(-2z^N)$ have the same number of zeros inside D_r, counted with multiplicities. Taking the limit $r \to 1$ from above yields N zeros of $P(z)$ inside the closed unit disk $D_{\bar{1}} = \{z : |z| \leqslant 1\}$, one of which is $z = 1$ (indeed, $P(1) = 1 - 2 + 1 = 0$). Let us check that no other roots lie on the boundary of the unit disk: the equation $0 = P(e^{i\theta}) = e^{iN\theta}(e^{i\theta} - 2) + 1$ yields, after rearranging and taking the norm, $|2 - e^{i\theta}| = 1$, which occurs only at $\theta = 0$. Therefore, the polynomials $P(z)$ and $p(z)$ have $N - 1$ roots $\lambda_1, \ldots, \lambda_{N-1}$ inside the open unit disk $D_1 = \{z : |z| < 1\}$, leaving one (real) root of $P(z)$ and $p(z)$ outside the closed unit disk.

2. Now we bound this big real root. Restrict $P(x)$ to $x \in [1, 2] \subsetneq \mathbb{R}$, where $P(1) = 0$, $P(2) = 1$, and $P'(x) = x^{N-1}\bigl((N+1)x - 2N\bigr)$. As $P'(1) < 0$, the function $P(x)$ is negative over some interval $(1, 1+\varepsilon)$ close to 1. The extrema of $P(x)$ are at $x_1 = 0$ and $x_2 = 2N/(N+1) > 1$. As there is only one extremum in $(1, 2)$, it must be a minimum. By the continuity of

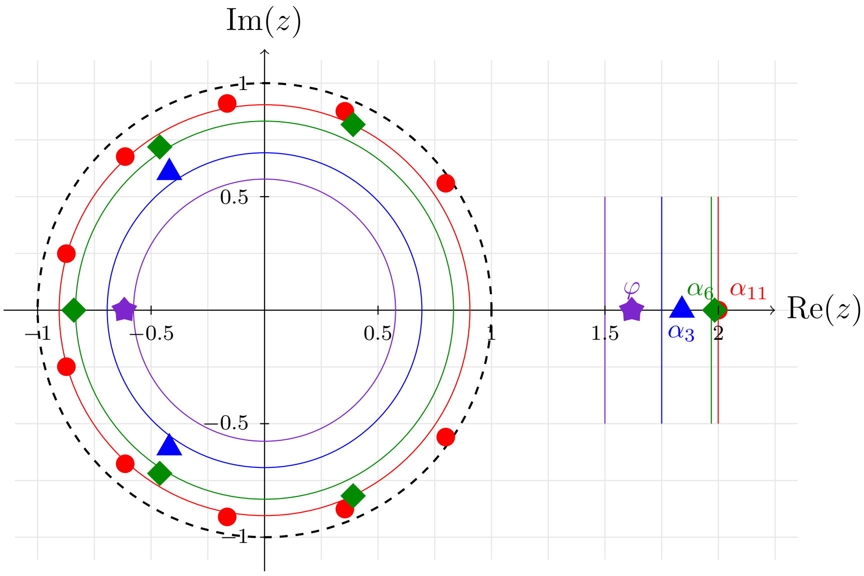

FIGURE 1. Real and complex roots of the characteristic polynomial for the N-bonacci numbers $p(z) = z^N - z^{N-1} - \cdots - z - 1$, plotted for the values $N = 2$ (Fibonacci numbers, golden ratio $\varphi \approx 1.61$, as **purple stars**), $N = 3$ (tribonacci numbers, $\alpha_3 \approx 1.84$, as **blue triangles**), $N = 6$ ($\alpha_6 \approx 1.984$, as **green rhombi**), and $N = 11$ ($\alpha_{11} \approx 1.9995$, as **red circles**). Shown as circles and straight lines are the lower bounds $\ell_N = 3^{-1/N}$ and $L_N = 2(1 - 2^{-N})$ for the complex modulus of the roots $\lambda_1, \ldots, \lambda_{N-1}$ in the complex unit disk and the positive real root α_N, respectively; the lower bounds for the complex roots' moduli are $\ell_2 \approx 0.577$, $\ell_3 \approx 0.693$, $\ell_6 \approx 0.832$, and $\ell_{11} \approx 0.904$, whereas for the positive real root the lower bounds are: $L_2 = 1.5$, $L_3 = 1.75$, $L_6 \approx 1.968$, and $L_{11} \approx 1.99902$.

the polynomial $P(x)$, there is a unique root $\alpha_N \in (x_2, 2) \subsetneq (1, 2)$ of $P(x)$ (and of $p(x)$). As the second derivative $P''(x)$ vanishes only at $x = 0$ and $x = x_2 - 2/(n + 1) < x_2$, it follows that $P(x)$ is convex up over $(x_2, +\infty) \supsetneq (x_2, 2]$. Thus the root α_N must lie to the right of the intersection of the x-axis and straight line connecting the points $(x_2, P(x_2))$ and $(2, P(2))$ on the plane. This line is given by the formula $y(x) = P(x_2) + \bigl(1 - P(x_2)\bigr)(x - x_2)/(2 - x_2)$; the function $y(x)$ has the root at

$$x_3 = x_2 + (2 - x_2)/\bigl(1 - 1/P(x_2)\bigr) = 2 - x_2^{-N} > x_2 = 2N/(N + 1),$$

where the second (short) expression of x_3 is obtained by simplification. Hence $\alpha_N \in (x_3, 2) \subsetneq (x_2, 2) \subsetneq (1, 2) \subsetneq \mathbb{R}$ and $\lim_{N\to+\infty} \alpha_N = 2$.

Writing the closed-form solution for the recurrence $F_n^{(N)} = C\alpha_N^n + \sum_{j=1}^{N-1} c_j\lambda_j^n$ with $C > 0$ (and $c_j, \lambda_j \in \mathbb{C}$) completes the proof of the proposition. □

Remark 4. Write $\alpha_N = 2(1 - \varepsilon_N)$, where $\varepsilon_N \in (0, 1 - x_3/2) = (0, x_2^{-N}/2)$. Let us compare this upper bound for ε_N with Wolfram's [10, Lemma 3.6]: $\varepsilon_N \in (0, 2^{-N})$. Expanding $x_2^{-N}/2$ yields $\delta_N := x_2^{-N}/2 = \frac{1}{2}\bigl(1 + 1/N\bigr)^N 2^{-N}$, hence $\lim_{N\to+\infty}(\delta_N/2^{-N}) = e/2 \approx 1.359$. So, both the bounds are asymptotically proportional to each other and decay exponentially fast.

Back matter

Acknowledgements

First and foremost, I thank my mathematics supervisor Arthemy Kiselev for introducing and guiding me through the topic and associated mathematics more broadly, for the countless hours spent together planning, writing, and revising our manuscripts and preprints, and for supporting the project more generally; also for the wonderful ideas, anecdotes and stories, and life/career (incl. teaching) advice. I also thank my physics supervisor Maxim Mostovoy for agreeing to supervise the physical aspects of this work and for the extremely useful and illuminating discussions. I express my thanks and gratitude to my Honours College supervisor Thomas Jansen for providing facilities within the University of Groningen for me to work and study in, and his support throughout the last 2.5 years of study.

Special thanks to fellow student Kārlis Driba for proofreading parts of this thesis and feedback. Many thanks to friends and colleagues for useful and fun discussions and attending my talk and defence, and for their questions. Finally, I wish to thank my parents and family for all their continued support.

Anotācija

Anotācija. Lī struktūras ir neatņemamas algebriskās struktūras fizikā: iekavu operators mēra, vai divas simetrijas vai vektorlauku trajektorijas ir savstarpēji (ne)komutatīvas. Noteiktās kvantu lauku teorijās (piem., topoloģiskajā Janga–Milsa teorijā) Lī struktūras deformē par augstākām (homotopiskām) Lī struktūrām, ar kurām raksturo kolektīvas mijiedarbības. Ar Vronska determinantu var izveidot augstākas Lī struktūras: $\mathfrak{sl}(2)$ no vektoru laukiem ar polinomiem kā koeficientiem, kā arī tās deformācijas uz reālo skaitļu ass. Topoloģiskas konstrukcijas no cietvielu fizikas un kvantu hromodinamikas, proti, Skirmionu topoloģisko lādiņu, var uzrakstīt Dekarta koordinātās, izmantojot Vronska determinantus d dimensijās, ar kuriem arī var pārnest Lī struktūras uz augstākām dimensijām. Sekojot Kirilova, Kontsēviča un Moļeva darbiem, šīs disertācijas mērķis ir raksturot, kā aug N-ārās Lī algebras, kuras konstruētas no polinomiem ar d dimensionālu Vronska determinantu kā iekavu, atkārtoti tās elementus liekot iekavā: precīzāk, kuras algebras neaug vispār (to dimensija ir galīga)?

Mēs atbildam uz šo jautājumu, atrodot jaunas determinantu sakarības; lai tās pamanītu un hipotezētu, ka tās ir spēkā vienmēr, mēs izmantojam datoralgebras programmatūru. Pieļaujot nelielus pieņēmumus par to struktūru, mēs atrodam *visas* galīgi dimensionālās polinomu N-ārās Lī algebras ar Vronska determinantiem d dimensijās kā iekavām; visas šīs algebras līdzinās $\mathfrak{sl}(2)$. Mēs atradām arī Vita algebras (no konformālās kvantu lauku teorijas) stipro homotopiju deformācijas, kā arī vispārinājumus d kompleksās dimensijās. Visbeidzot, kad algebra var augt, mēs nosakam augšējo robežu asimptotiskai polinomu pakāpju augšanai; šī robeža sastāv no N-bonači skaitļiem.

Mēs izvirzām 10 atvērtus jautājumus, kas tieši izriet no mūsu rezultātiem – pirmajiem soļiem lielāku jautājumu atrisināšanai matemātiskajā fizikā. Mūsu izstrādātās struktūras ir paredzētas kā matemātiski rīki nākamās paaudzes modeļu izstrādei kalibrācijas lauku teorijā, it īpaši, lai pētītu kolektīvas mijiedarbības izpausmes un topoloģiskas īpašības.

SAMENVATTING

SAMENVATTING. Lie-structuren vormen fundamentele algebraïsche instrumenten in de natuurkunde: de Lie-haak meet de (niet-)commutativiteit van twee symmetrieën of van twee door vectorvelden geïnduceerde stromen. In bepaalde kwantumveldentheorieën, zoals de topologische Yang–Mills-theorie, worden deze gedeformeerd tot hogere (homotopische) Lie-structuren, waarbij haken van hogere ariteit collectieve interacties coderen. Wronski-determinanten realiseren hogere Lie-structuren, waaronder $\mathfrak{sl}(2)$ via vectorvelden met polynomiale coëfficiënten, en de bijbehorende deformaties op de reële lijn. Ook topologische constructies uit de gecondenseerde-materiefysica en de kwantumchromodynamica – de topologische lading van Skyrmionen – worden in cartesische coördinaten gerealiseerd door Wronski-determinanten op $\mathbb{R}^d$, waarmee Lie-structuren tevens naar hogere basisdimensie worden gedeformeerd. In navolging van Kirillov, Kontsevich en Molev luidt het probleem: beschrijf de groei onder iteratie van de haak in N-aire Lie-algebra's, opgebouwd uit polynomen, met Wronski-determinanten op $\mathbb{R}^d$ als haak; in het bijzonder, welke zijn alle eindig-dimensionale algebra's (die onder iteratie helemaal niet groeien)?

Wij beantwoorden deze vraag door nieuwe determinantidentiteiten te bewijzen. Om deze identiteiten te ontdekken en te vermoeden dat zij altijd gelden, maken wij gebruik van computeralgebra. Onder milde aannames over hun structuur classificeren wij alle eindig-dimensionale polynomiale N-aire Lie-algebra's met Wronski-determinanten op $\mathbb{R}^d$ als haken; zij zijn alle gemodelleerd naar $\mathfrak{sl}(2)$. Daarnaast verkrijgen wij sterke homotopiedeformaties van de Witt-algebra uit de conforme veldentheorie over $\mathbb{C}$ en veralgemeningen daarvan op $\mathbb{C}^d$. Ten slotte bepalen wij, in de gevallen waarin de algebra wel kan groeien, een asymptotische bovengrens voor de groei van de polynomiale graad; deze wordt gegeven door de N-bonaccigetallen.

Deze resultaten vormen eerste stappen naar bredere vraagstukken binnen de mathematische fysica, en de tien problemen die wij vervolgens formuleren, staan onmiddellijk voorbij deze resultaten. Wat wij hier ontwikkelen is bedoeld als wiskundige hulpmiddelen voor modellen van de volgende generatie in de ijkveldentheorie, in het bijzonder voor de studie van collectieve effecten en topologische eigenschappen.

SUMMARY FOR LAYMEN

The mathematics of modern physics consists of, to a large degree, differential equations and algebraic constructions to aid in solving them (which is a very difficult task). Much of XIX and early-to-mid XX century physics uses Lie structures: for instance, equations of motion in classical mechanics can be written with a Poisson bracket (of two smooth functions as arguments), quantum mechanics is built using commutators, chiefly of position and momentum (operators), and the structures in quantum field theory are described using generators of symmetries (e.g., rotations) in Lie & Clifford algebras.

In very high-level theoretical physics Lie structures are not sufficient for an accurate description, and they are deformed to 'higher Lie' structures to fit the needs. It is a miracle that using elementary constructions from 1$^{\text{st}}$ year mathematics – polynomials and their derivatives plugged into determinants (so-called Wronskian determinants) – one is able to create such higher Lie structures, in particular, the Lie algebra $\mathfrak{sl}(2)$ of (Einstein's) special relativity (more precisely, the Lorentz group).

Back in the early 2000s only two examples of (finite-dimensional) deformations of $\mathfrak{sl}(2)$ were known. We have now found infinitely many such examples: they exist for all parameter values (base dimension and highest-appearing derivative) and are all of a very specific form. We also found deformations and generalisations of another Lie algebra from theoretical physics – the Witt algebra.

We have done more work on the growth of higher Lie algebras; it is a question of mathematical physics dating back to the early 1980's by Kirillov: how fast do they grow (if at all)? Kirillov and co-authors answered this question in the case of purely Lie structures. We extend his problem from the Lie case to the higher Lie case and prove an upper bound for the rate of their growth.

Finally, we have found connections between physically real, useful, and highly applicable constructions in condensed matter physics – the Skyrmions – and the Wronskian determinants. The ease of writing more general Wronskians opens the doors to searching for higher topological invariants in condensed matter physics and beyond.

What we did are base steps towards solving bigger problems in mathematical physics. We have formulated 10 open problems, which stand immediately beyond our results, and advance closer to the problems. These serve as good avenues for future research.

Markuss G. Ķēniņš

Contact:	mkenins@student.ethz.ch
	m.g.kenins@student.rug.nl
	markuss.kenins@gmail.com
Webpage:	https://www.mgk.ac
Phone:	+371 29395183

10 July, 2026
Groningen, the Netherlands