ANALYSIS OF VESSEL DEPRESSURISATION IN VACUUM

TEAM 374

Abstract. Problem **A** was chosen and solved. A punctured chamber pressurised to 1 atmosphere surrounded by vacuum was analysed for gas evacuation leading to a pressure drop; three models for evacuation time to 0.3 atmospheres were developed and analysed. It was shown that an ideal evacuation process is isothermal. A model for mass flow rate through a small duct was used; assuming conditions of a perfect gas, the time to reach 0.3 atmospheres was computed in limiting isentropic and isothermal cases, which gave bounds of 10.5 h and 13.3 h, respectively, which gives a solution to the problem. The results were compared to the value obtained in a NASA report; the predictions from the model agree with the reported values. The model allowed to obtain the evacuation time dependence upon the hole diameter, which was found to be an inverse square law. In the maximal hole limit, a lower bound using the speed of sound was computed for evacuation, obtaining 0.146 s; our results were found to be above this bound. A third model of effusion developed in the small hole limit was shown to not be applicable and produced strange results for a 1 cm diameter hole.

Contents

1. Introduction

In the near future, Humanity will expand its habitat from a single planet to its neighbourhood. In this extraterrestrial journey, a big threat is posed by the micrometeorites, which can puncture through the walls of the spaceships and subsequently decompress the cabins. Various studies conducted by NASA using chimpanzees and other subjects have proven that a complete lack of pressure significantly lowers the life expectancy of the subjects [2].

In order to estimate the available time to fix the hole before a significant pressure drop, the group carried out the following comprehensive research on the process of decompression. A space station in our model is represented by a cylinder with an interior length of 50 meters and a diameter of 4 meters (see Fig. 1). The air inside the space station at time $t = 0$ is at a temperature of $T = 293\,\mathrm{K}$ and an atmospheric pressure ($P_i = 101.3\,\mathrm{kPa}$). A micrometeorite makes a hole at $t = 0$ with a diameter of $d = 1\,\mathrm{cm}$ in the centre of one end of the space station. The goal is to find how much time would it take until the air pressure in the space station was only 0.3 of an atmosphere. And how would this time be different for holes of different diameters?

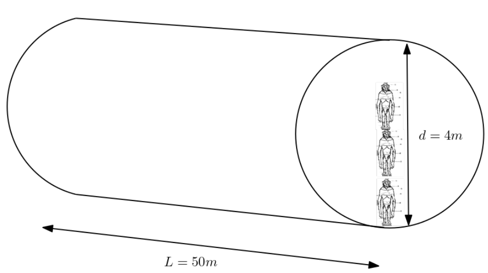

Figure 1. The station with a chimpanzee (from [2]) for scale

We will perform analysis first in the limit of a very small home; then we will derive a model for any sized hole and obtain the dependence on the hole diameter d. We will separately derive the case when the hole is the whole size of the cylinder (open tube); this will be done by using speed of sound to obtain the maximal velocity in this case, after which we will check whether all our results are no smaller than this lower bound. Throughout the problem an ideal gas will be assumed, and that there is no turbulence within the cylinder – temperature and pressure may be described using average values at any length along the cylinder.

2. Theoretical description

2.1. **Expansion against vacuum.** First, some approximations to simplify the problem are introduced. Let the gas (air) be ideal with diatomic molecules of molar mass $M \approx 29\,\mathrm{gmol}^{-1}$ as it consists of mainly Nitrogen and Oxygen. The space station is a cylinder of length L and diameter d_{cyl}, where x is the coordinate from the back wall ($x = 0$) until the wall with a hole of diameter d ($x = L$). The setup is symmetric around the cylinder's axis, removing axial dependence. Next, let $n(t)$ denote the number of air molecules per unit volume inside the cylinder at any time t at temperature $T(x,t)$ and pressure $p(x,t)$. Given the cylindrical symmetry of the cylinder, there is no dependence upon the angle of rotation; T and p may depend on the distance from the central x-axis. In our solution, we will ignore such dependence.

First, it is shown that the temperature inside the cylinder remains constant; this is done by considering work. The system is taken to be the spaceship together with the surrounding vacuum; no heat is exchanged via radiation and since the vacuum contains no particles, no exchange of heat is possible, hence it is an isolated system with zero heat change $dQ = 0$. The external pressure resisting the movement of gas is zero $p_{\mathrm{ext}} = 0$ since the gas expands against vacuum. Then the work dW done on the gas, given by $dW = -p_{\mathrm{ext}}dV$, is $dW = 0$. Now, the First Law of Thermodynamics states that the change in internal energy of a gas is given precisely as the sum of the change in heat and in work $dU = dQ + dW$; since both $dQ = dW = 0$, it follows that $dU = 0$. Now, since $dT \propto dU$, it follows that $dT = 0$, and thus there is no change of temperature of the gas – the expansion is isothermal.

A real gas will likely experience friction against the walls of the cylinder at the opening and with itself for velocities high enough to make laminar flow impossible.

2.2. **Speed of sound: The lower limit.** As a rough sanity check, it is proposed to use the following consideration: the shock-wave in gas inside the station, during the expansion (even if the hole diameter is large enough for the explosive decompression), cannot move faster than the speed of sound in the respective medium (i.e. the inside of the decompressing space station), since a rapid decompression is (to the first order) an adiabatic process, it is assumed that $\Delta U = 0$, which allows one to assume that the temperature of the gas remains constant. Knowing the fact that the speed of sound in an ideal gas is given by $v_s = \sqrt{\frac{\gamma RT}{M}}$, it is clear that in an event of the rapid decompression, it can be considered constant and equal to the $v_s = 343\,\mathrm{m/s}$. It is then possible to calculate a **rough** lower bound for the *complete* decompression time of the station. This limit is then taken as $t_{min} = \frac{L}{v_s} = \frac{50\,\mathrm{m}}{343\,\mathrm{m/s}} = 0.146\,\mathrm{s}$ since this is the time it takes for the farthest shock-wave(starting at the other side of the space station) to reach the hole. As was said earlier this is the lowest limit of the complete decompression, if any result for complete/partial decompression is lower than this one it can be considered unphysical.

2.3. **Isothermal effusion.** The first crude solution attempt used a constant temperature of the gas and effusion mode. This is not accurate as the mean free path of air molecules λ is much smaller than d, the hole diameter $\lambda = k_B T/\sqrt{2} p\sigma \approx 6\,\mu\mathrm{m} \ll d = 1\,\mathrm{cm}$, where collision cross-section of air $\sigma \approx 0.42\,\mathrm{nm}^2$ was used [3].

Regardless, in this approximation, the particle flux per unit area is $\psi = \frac{1}{4} n \langle v \rangle$([5]) obtained by integrating the number of molecules hitting a surface at angle θ with velocity v over all possible velocities v and angles and assuming the Maxwell–Boltzmann distribution. This also gives $\langle v \rangle = \sqrt{8 k_B T/\pi m}$ with m being a mass of a single air molecule. The rate of gas molecules leaving the station is given by: $\dot{N} = -A\phi$. Assuming uniform air concentration over the space station volume V_s, $\dot{N} = V_s \dot{n}$. Then the following is obtained by combining everything:

$$\dot{n} = -\sqrt{\frac{\pi k_B T}{32 m}} \frac{d^2}{V_s} n \tag{2.1}$$

From the ideal gas law at a constant temperature, $p \propto n$, this equation also governs the pressure time evolution from the initial p_0, given by:

$$p(t) = p_0 \exp\left\{ -\sqrt{\frac{\pi k_B T}{32 m}} \frac{d^2}{V_s} t \right\} \tag{2.2}$$

This also assumes the pressure is homogeneously changing throughout the whole station, which makes sense only for slow evacuation. The time to reach a pressure of $0.3 p_0$ is solved easily as:

$$t = \sqrt{\frac{32 m}{\pi k_B T}} \frac{V_s}{d^2} \ln \frac{10}{3} \approx \frac{0.848}{d^2} \tag{2.3}$$

This suggests an inverse square dependence of the evacuation time on the hole diameter. For example for $d = 1\,\mathrm{cm}$: $t \approx 8400 s \approx 2\,\mathrm{h} 20\,\mathrm{min}$ and for the maximum sized hole $d = 4\,\mathrm{m}$: $t \approx 0.054\,\mathrm{s}$(see Fig. 2).

2.4. **Derivation of the expansion of a perfect gas.** The area of the small hole is $A = \pi d^2/4$, whereas the volume of the cylinder $V = (\pi/4) d_{\mathrm{cyl}}^2 L$. Here it is assumed that the expansion is perfectly isothermal and the gas is perfect (thus ideal), namely, that $pV = mR'T$, where we let R' be R over the molar mass of air. We shall adopt the treatment from [6]. Upon differentiation, $V\dot{p} = mR'\dot{T} + R'T\dot{m}$, where n_m is the number of moles of particles in V.[1]

2.4.1. *Isentropic flow.* For isentropic flow, $pV^\gamma = C$ constant allows to find

$$\frac{dT}{dt} = \frac{\gamma - 1}{\gamma} \frac{T}{p} \frac{dp}{dt}. \tag{2.4}$$

The mass flux $G = -\rho V$ allows to find the mass flow rate $\dot{m} = \rho A V$. Now, the density $\rho = p/R'T$ and speed of sound $c = \sqrt{\gamma R' T}$ allows us to write $G =$

[1]Note that in [6] the notation is different – R is given over the average molar mass of air, here we use R'.

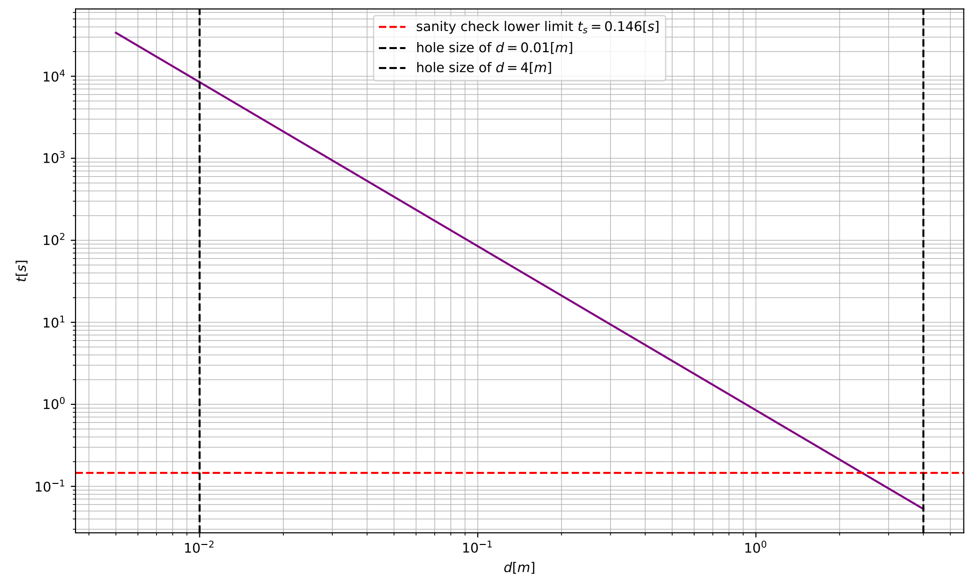

FIGURE 2. Effusion Model of decompression

$p(V/c)\sqrt{\gamma/RT}$; we use the Mach number M of flow velocity over c to obtain

$$G = -\frac{p_0}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R'}}M\left[1+\frac{1}{2}(\gamma-1)M^2\right]^{-\frac{\gamma+1}{2|\gamma-1|}}, \tag{2.5}$$

which attains a minimum when $M = 1$ (see [6]), namely,

$$G = -\frac{p_0}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R'}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{\gamma-1}}}. \tag{2.6}$$

Now, this allows us to write

$$V\dot{p} = mR'\frac{\gamma-1}{\gamma}\frac{T}{p}\frac{dp}{dt} - R'T\frac{p_0 A}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R'}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{\gamma-1}}}, \tag{2.7}$$

which upon heavy rearrangement gives $\dot{p} = -Cp^{(3\gamma-1)/2\gamma}$, where the remaining expression is grouped into the constant

$$C = \frac{\gamma}{V}\frac{R'\sqrt{T_0}A}{p_0^{(\gamma-1)/2\gamma}}\sqrt{\frac{\gamma}{R'}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{\gamma-1}}}. \tag{2.8}$$

The differential equation is separable, hence we solve it by integrating both sides. We obtain the pressure p at time t_f by integrating $\dot{p}$ from p_0 to $p_f = (3/10)p_0$ and integrating C from $t = 0$ to $t = t_f$, namely, $\int_{p_0}^{p_f} p^{-(3\gamma-1)/2\gamma}\, dp = -C\int_0^{t_f} dt$. Upon straightforward integration and rearrangement, this gives

$$t_f = -\frac{1}{C}\frac{2\gamma}{1-\gamma}p_0^{(1-\gamma)/2\gamma}\left[\left(\frac{p_f}{p_0}\right)^{\frac{1-\gamma}{2\gamma}} - 1\right]. \tag{2.9}$$

We have thus obtained the theoretical solution for an ideal gas with a specific heat ratio γ at isothermal conditions. Inserting the value $\gamma = 1.4$ for air, we obtain

$$t_{f,I} = \frac{0.43V_s}{A\sqrt{T_i}}\left[\left(\frac{p_i}{p_f}\right)^{0.143} - 1\right] \tag{2.10}$$

the time to reach p_f from p_0 under isentropic conditions.

2.4.2. *Isothermal flow.* Now, we change the assumption from isentropic to isothermal expansion. Temperature being constant gives the time derivative of the ideal gas law to be $V\dot{p} = \dot{m}R'T$; upon similar substitution of $\dot{m}$ from Eq. (2.6), we obtain that $\dot{p} = C'p$, where the constant is

$$C' = -\frac{A}{\sqrt{T_0}}\sqrt{\frac{\gamma}{R'}\left[\frac{2}{\gamma+1}\right]^{\frac{\gamma+1}{\gamma-1}}}\,\frac{R'T}{V}. \tag{2.11}$$

As before, we have a separable equation for $\dot{p}$, which integrated from $t = 0, p = p_0$ to $t = t_f, p_f$ gives the prediction for the time to reach p_f from p_0 to be

$$t_f = \frac{1}{C'}\ln\frac{p_f}{p_0}. \tag{2.12}$$

Again, with the value for air $\gamma = 1.4$ inserted, the formula above becomes

$$t_{f,T} = \frac{0.086V_s\ln\left(\frac{p_i}{p_f}\right)}{A\sqrt{T_i}}. \tag{2.13}$$

The equations for $t_{f,I}$ and $t_{f,T}$ will be used for our final computations in the next subsection.

2.5. **Computation of the final result.** The two extreme cases, namely, time to reach $(3/10)p_0$ under isentropic conditions $t_{f,I}$ from Eq. (2.10) and isothermal $t_{f,T}$ from Eq. (2.13), impose bounds on the real evacuation time and both notably depend on the inverse square of the hole diameter. More specifically, plugging in the values in of this problem:

$$t_{f,I} = \frac{3.78}{d^2} < t < \frac{4.84}{d^2} = t_{f,T} \tag{2.14}$$

the plot of these limits is given there Fig. 3. The values for $d = 0.01\,\mathrm{m}$ are $37\,800\,\mathrm{s} < t_{f,0.01} < 48\,400\,\mathrm{s}$ and for $d = 4\,\mathrm{m}$ are $0.24\,\mathrm{s} < t_{f,4} < 0.30\,\mathrm{s}$, both of the times above are larger than the lower limit imposed by the speed of sound (passes sanity check).

3. Results and discussion

3.1. **Effusion.** As it was said in the beginning, the effusion model is the crudest approximation of the physical process in question. That is simply because the fundamental condition for the effusion i.e. that the mean free path of the molecules λ should be of the same order of magnitude as the hole through which they effuse, is broken. It is not surprising therefore that the results obtained from this model are far from realistic, which can be observed by simply noting that the decompression time is actually able to cross the sanity limit of $t_{\min} = 0.146\,\mathrm{s}$ as defined in Section 2.2, which can be observed on the graph Fig. 2. On the other hand, it is worth acknowledging that such a model would be sufficient for explaining the

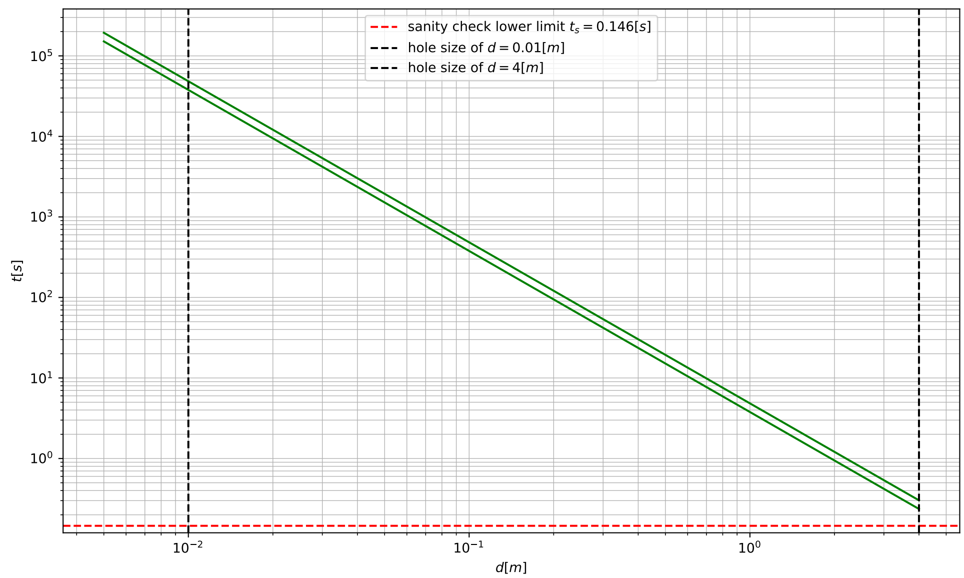

FIGURE 3. Saad's model of decompression (upper and lower limits)

effects of the *cracks* in the hull of the space station, which(unlike *meteoroids*) can be of micrometre scale in width.

3.2. **Main result.** The most versatile model was the one derived from Euler's equations, giving the range of possible time values, which is more valuable than a single value based on a particular set of assumptions. In the isentropic case, no heat interactions take place, while in the isothermal case, the maximum heat interaction occurs to keep the temperature constant. The real value is then expected to lie in between those physically opposite extreme assumptions. The scaling of the time inversely with hole area, in our case diameter squared, also makes intuitive sense on the interval where a single model is used.

The results obtained are very close to the ones that NASA writes in its internal documentation for the ISS [1]. Those mention 14 hours to depressurise $900\,\mathrm{m}^3$ to 9.5psi $= 0.646\,\mathrm{Pa}$ from a $0.6\,\mathrm{cm}$ hole. Plugging those values into our model (Eq. (2.10),Eq. (2.13)) yields a time interval prediction of $14\,\mathrm{h}20\,\mathrm{min} < t < 19\,\mathrm{h}30\,\mathrm{min}$. This is very close to the official result and justifies our assumption as practical.

However, for large hole diameters, the concentration and pressure can no longer be taken as homogenous and effusion can definitely not be used either. In any model in general, problems will occur at the boundary which has an initial discontinuity and will result in behaviour not well describable by Euler and Navier-Stokes equations, requiring molecular models beyond analytical capacity. The book [4] includes a similar problem in chapter 13 with gas outside and vacuum inside a cylinder, and the molecular simulation of the behaviour is very complex indeed. Lastly, air was always taken to be an ideal gas which could use further refinement.

3.3. **Failed attempts.** The first one used the effusion equation Appendix A with a temperature depending on concentration. This resulted in $p \propto n^2$ from the ideal gas law, which caused a significantly slower pressure decrease for the same concentration decrease. The depressurization timescale got stretched and became unreasonable on the order of days for a centimetre hole.

Another failed approach we considered was trying to simulate the gas motion in the station itself by adding together our discovered Eq. (2.6), and the diffusion equation. The system we got was solved numerically using a simulation, it was found that the diffusivity of the gas is so small, that it does not account for any meaningful motion of the gas and does not represent any physically significant process in our case.

Another approach was considering an adiabatic expansion, where the volume at a time t was taken to be the station volume plus the volume of an expanding hemisphere of the escaped gas. However, defining the volume and temperature of this gas escaping rapidly at different speeds is tricky. The results are, that even if the hemisphere was expanding at 3 meters per second, or 1% of the speed of sound, the depressurization time is less than 0.5 seconds for the centimetre hole.

The last failed attempt again used Eq. (2.6), together with fast molecules disproportionately leaving the station according to Maxwell–Boltzmann distribution with an additional velocity factor. Then the rate of change of total energy of the gas inside the station was expressed both using a change of temperature and a change of the number of those fast molecules. The temperature was assumed to be changing, but homogenous throughout the station along with pressure and concentration, giving a final result of the form $p(t) = \frac{1}{(at+b)^8}$. This result predicted depressurization in the millisecond range for the centimetre hole, making it implausible.

4. Conclusions

The following have been concluded to be true.

(1) Under ideal conditions the expansion into a vacuum is isothermal.
(2) A model for gas escaping through a nozzle was found to be applicable; an analytical solution may be found.
(3) Isentropic and isothermal conditions give bounds for the evacuation time from 1 to 0.3 atmospheres; the bounds obtained are 10.5 and 13.3 hours, respectively. The model agrees with values from a NASA report about ISS [1] predicting 14.3-20 hours whereas NASA predicts 14 hours.
(4) The depressurization time dependence on hole diameter follows an inverse square law.
(5) For any size of the hole, a lower bound is given by gas escaping at the speed of sound; this limit is 0.146 s.

Appendix A. Non-isothermal attempt

Here an attempt is detailed, where the postulate of isothermal expansion is false. We will consider the small hole limit, where the temperature T is the same throughout the cylinder and is proportional to the number of air molecules per volume $T = \alpha^2 n$ with α^2 being the proportionality constant in $\mathrm{K/m^3}$. This specific relationship $T = \alpha^2 n$ of the form $T = f(n)$ was tested because other relationships of the form $T \propto n^\ell$ for some power ℓ are likely to produce qualitatively similar results; more complicated relationships are not likely to easily give an analytical solution.

We begin by inserting T into Eq. (2.1), obtaining

$$\frac{dn}{dt} = -\sqrt{\frac{\pi k_B}{32m}}\frac{d^2\alpha}{V_s}n^{3/2} = -\omega\alpha n^{3/2}, \tag{A.1}$$

where we set $\omega = (d^2/V)\sqrt{\pi k_B/32m}$. It may be easily verified that the solution to the above equation is given by

$$n(t) = \frac{4}{[\omega\alpha t + c]^2} \tag{A.2}$$

for some free constant c. Upon imposing the condition that n at $t = 0$ attains the value with the hole closed $n(0) = p_0/kT_0$, we obtain that $c = 2\sqrt{kT_0/p_0}$; the constant α is found from $T_0 = T(0) = \alpha^2 n(0) = \alpha^2 p_0/kT_0$, thus $\alpha = \sqrt{T_0}\sqrt{kT_0/p_0}$, from which it follows that

$$n(t) = 4\frac{p_0}{kT_0}\left[\sqrt{T_0}\omega t + 2\right]^{-2}. \tag{A.3}$$

Pressure is obtained from the ideal gas law as $p(t) = n(t)kT(t) = \alpha^2 kn^2$. We find the ratio of the initial pressure $p_0 = 1\,\mathrm{atm}$ over the final $p(t_f) = (3/10)p_0$ to be

$$\frac{3}{10} = \frac{\alpha^2 kn^2(t_f)}{\alpha^2 kn^2(0)} = 16[\sqrt{T_0}\omega t_f + 2]^{-4} \tag{A.4}$$

which gives

$$t_f = \frac{1}{\sqrt{T_0}\omega}\left[\left(\frac{160}{3}\right)^{1/4} - 2\right] \approx 768\,709\,\mathrm{s} \approx 9\,\mathrm{days}. \tag{A.5}$$

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